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# Finding Energy of the Slant Helix Strip by Using Classic Energy Methods on Joachimsthal Theorem

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**Abstract:** There are two forms of mechanical energy-potential energy and kinetic energy in physics. Potential energy $E_p$ is stored energy of position. The amount of kinetic energy $E_k$ possesed by a moving object is depent upon mass and speed. The total mechanical energy possesed by an object is the sum of its kinetic and potential energies. Now we calculate the mathematical physic on Joachimsthal Theorem. In this paper, we find the eneryg of two curves on different surfaces and slant helix strips by using classic energy formulaes in Euclidean Space $E^3$.

**Keywords:** Curve-Surface Pair (Strip), Curvature, Energy, Classic Energy Formulaes, Joachimsthal Theorem

## 1. Introduction

Now we look what we know about the terms of the energy.

The word ‘energy’ comes from energeia in Greek. First occurred in the studies of Aristoteles in 4th century B. C.

The ‘energy’ term came from by defining Gottfried Leibniz that was vis viva (live force). Leibniz defined vis viva that is the multiplying matter's mass and its squared velocity.

In 1807, Thomas Young used energy term as meaning of today instead of vis visa that was the first person. Gustave-Gaspard Coriolis defined kinetic energy in 1829; William Rankine defined potential energy in 1853 as todays meanings.

Energy is not only used by physicists but also by mathematicians.

We know that there are two forms of mechanical energy-potential energy and kinetic energy. Potential energy is stored energy of position. Such energy is known as the gravitational energy $E_p$ and is calculated as

$$E_p = m.g.h$$

where $m$ is the mass of the object, $g$ is the acceleration of gravity (9,8m/s) and $h$ is the height of the object (with standart units of meters). Kinetic energy is defined as the energy possesed by an object due its motion. An object must be moving to possess kinetic energy. The amount of kinetic energy $E_k$ possesed by a moving object is depent upon mass and speed. The equation for kinetic energy is

$$E_k = \frac{1}{2} m.V^2$$

where $m$ is the mass of the object (with standard onits of kilograms) and $V$ is the speed of the object (with standart units of m/s). The total mechanical energy possesed by an object is the sum of its kinetic and potential energies as:

$$E_{T=}E_p + E_k \text{ [13].}$$

In this paper, we find the eneryg of two curves on different surfaces and slant helix strips by using classic energy formulaes in Euclidean Space. But a new paper may be studied with the same conditions in Lorentzian Space $E_1^{n+1}$. See [14, 15].

## 2. Preliminaries

We now review some basic concepts on classical differential geometry of space curves in Euclidean space, general helix and slant helix. Let $\alpha : I \rightarrow R^3$ be a curve $\alpha'(s) \neq 0$ where $T(s) = \alpha'(s)$ is a unit tangent vector of $\alpha$ at s and $M$ be a surface in Euclidean 3-space. We define a surface element of $M$ is the part of a tangent plane at the neighbour of the point. The locus of the these surface element along the curve is called a curve-surface pair as shown $(\alpha, M)$. We study this Euclidean Space, may study in Minkowski space and rotational surfaces. See more details in [7, 13].

### *2.1. The Curve-Surface Pair (Strip)*

*Definition:*

Let $M$ and $\alpha$ be a surface in $E^3$ and a curve in $M \subset E^3$. We define a surface element of $M$ is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve $\alpha$ is called a curve-surface pair and is shown as $(\alpha, M)$.

*Definition:*

Let $\{\vec{t}, \vec{n}, \vec{b}\}$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the curve and curve-surface pair's vector fields. The curve-surface pair's tangent vector field, normal vector field and binormal vector field is given by $\vec{t} = \vec{\xi}$, $\vec{\zeta} = \vec{N}$ $(\vec{N} = \vec{n})$ and $\vec{\eta} = \vec{\zeta} \wedge \vec{\xi}$ ([1-6, 8-10]).

### *2.2. Curvatures of the Curve-Surface Pair and Curvatures of the Curve*

Let $k_n = -b, k_g = c, t_r = a$ and $\{\vec{\xi}, \vec{\eta}, \vec{\zeta}\}$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip and the curve-surface pair's vector fields on $\alpha$ [1-6, 9, 10].

Then we have

$$\begin{aligned} \vec{\xi}\,' &= c\vec{\eta} - b\vec{\zeta} \\ \vec{\eta}\,' &= -c\vec{\xi} + a\vec{\zeta} \\ \vec{\zeta}\,' &= b\vec{\xi} - a\vec{\eta}. \end{aligned} \tag{1}$$

We know that a curve $\alpha$ has two curvatures $\kappa$ and $\tau$. A curve has a strip and a strip has three curvatures $k_n, k_g$ and $t_r$.

Let $k_n, k_g$ and $t_r$ be the $-b, c$ and a. From last equations we have $\vec{\xi}\,' = c\vec{\eta} - b\vec{\zeta}$. If we substitude $\vec{\xi} = \vec{t}$ in last equation, we obtain

$$\vec{\xi}\,' = \kappa\vec{n}$$

and

$$b = -\kappa \sin\varphi$$
$$c = \kappa \cos\varphi$$

([2-6, 9, 10]) From last two equations we obtain,

$$\kappa^2 = b^2 + c^2 .$$

This equation is a relation between the curvature $\kappa$ of a curve $\alpha$ and normal curvature and geodesic curvature of a curve-surface pair.

By using similar operations, we obtain a new equation as follows

$$\tau = a + \frac{b'c - bc'}{b^2 + c^2}$$

([2-6, 9, 10]). This equation is a relation between $\tau$ (torsion or second curvature of $\alpha$ and curvatures of a curve-surface pair that belongs to the curve $\alpha$ ). And also we can write

$$a = \varphi' + \tau .$$

*The special case:*

If $\varphi$ is constant, then $\varphi' = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

*Definition:*

Let $\alpha$ be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve $\alpha$ is equal to zero, then the curve-surface pair $(\alpha, M)$ is called a curvature curve-surface pair (strip) ([2-6, 9, 10]).

# 3. General Helix

*Definition:*

Let $\alpha$ be a curve in $E^3$ and $V_1$ be the first Frenet vector field of $\alpha$. $U \in \chi(E^3)$ be a constant unit vector field.

If

$$\langle V_1, U \rangle = \cos\varphi \text{ (constant)}$$

$\alpha$, $\varphi$ and $Sp\{U\}$ are called a general helix, the slope and the slope axis ([1, 2, 6]).

*Definition:*

A regular curve is called a general helix if its first and second curvatures $\kappa$ and $\tau$ are not constant but $\frac{\kappa}{\tau}$ is constant ([1, 6]).

*Definition:* A curve is called a general helix or cylindrical helix it its tangent makes a constant angle with a fixed line in space. A curve is a general helix if and only if the ratio $\frac{\kappa}{\tau}$ is constant ([5, 9, 12]).

*Definition:* A helix is a curve in 3-dimensional space. The following parametrisation in Cartesian coordinates defines a helix, see [7].

$$x(t) = \cos t$$
$$y(t) = \sin t$$
$$z(t) = t.$$

As the parameter $t$ increases $(x(t), y(t), z(t))$ traces a right-handed helix of pitch $2\pi$ and Radius 1 about the $z$ axis, in a right-handed coordinate system. In cylindrical coordinates $(r, \theta, h)$ the same helix is parametrised by

$$r(t) = 1,$$
$$\theta(t) = t,$$
$$h(t) = t.$$

*Definition:*

If the curve $\alpha$ is a general helix, the ratio of the first curvature of the curve to the torsion of the curve must be the constant. The ratio $\frac{\tau}{\kappa}$ is called first harmonic curvature of the curve and is denoted by $H_1$ or $H$.

*Theorem 3.1:* A regular curve $\alpha \subset E^3$ is a general helix if and only if $H(s) = \frac{k_1}{k_2} = const$ for $\forall s \in I$, see [7].

*Proof:* $(\Rightarrow)$ Let $\alpha$ be a general helix. The slope axis of the curve $\alpha$ is showed $Sp\{U\}$. Note that

$$\langle \alpha'(s), U \rangle = \cos\varphi = const.$$

If the Frenet Threshold is $V_1, V_2, V_3$ at the point $\alpha(s)$, then we have

$$\langle V_1(s), U \rangle = \cos\varphi.$$

If we take derivative of the both sides of the last equation, then we have

$$\langle k_1 V_2(s), U \rangle = 0 \Rightarrow \langle V_2(s), U \rangle = 0.$$

Hence

$$U \in Sp\{V_1(s), V_3(s)\}.$$

Therefore

$$U = \cos\varphi V_1(s) + \sin\phi V_3(s).$$

$U$ is the linear combination of $V_1(s)$ and $V_3(s)$. By differentiating the equation $\langle V_2(s), U \rangle = 0$, we obtain

$$\langle -k_1 V_1(s) + k_2 V_3(s), U \rangle = 0,$$
$$-k_1(s)\langle V_1(s), U \rangle + k_2(s)\langle V_3(s), U \rangle = 0,$$
$$-k_1(s)\cos\phi + -k_2(s)\sin\varphi = 0.$$

By using the last equation, we see that

$$H = const.$$

$(\Leftarrow)$ Let $H(s)$ be constant for $\forall s \in I$, and $\lambda = \tan\varphi$, then we obtain

$$U = \cos\varphi V_1(s) + \sin\varphi V_3(s).$$

1. If $U$ is a constant vector, then we have

$$D_{\dot{\alpha}} U = \left(k_1(s)\cos\varphi - \sin\varphi k_2(s)\right)V_2(s).$$

By substituting $H(s) = \tan\varphi$ is in the last equation, we see that

$$k_1(s)\cos\varphi - k_2 \sin\varphi = 0,$$

and so

$$U = const.$$

2. If $\alpha$ is an inclined curve with the slope axis $Sp\{U\}$, then

$$\langle \alpha'(s), U \rangle = \langle V_1(s), \cos\varphi V_1(s) + \sin\varphi V_3(s) \rangle$$
$$= \cos\varphi \langle V_1(s), V_1(s) \rangle + \sin\varphi \langle V_1(s), V_3(s) \rangle,$$

and we obtain

$$\langle \alpha'(s), U \rangle = \cos\varphi = const \text{ ([7])}.$$

Definition:

Let $S^2$ and $\alpha$ be a sphere in $E^3$ and a helix that lies on the sphere $S^2$. The curve $\alpha$ is called a spherical helix which lie on the sphere [12].

Definition:

Let $\alpha$ be a helix in $M \subset E^3$. We define a surface element of $M$ is the part of a tangent plane at the neighbour of the point of the helix that lie on $M$. Instead of the geometric plane of these surface elements along the helix $\alpha$ which lie sphere $M$ is called a helix strip.

Definition:

Let $S^2$ be a sphere and and $\alpha$ a helix which lie on $S^2$ in $E^3$. We define a surface element $S^2$ is the part of a tangent plane at the neighbour of the point of the helix that lie on $S^2$. The locus of these surface elements along the helix $\alpha$ which lie on the sphere $S^2$ is called spherical helix strip.

# 4. Finding Energy of the Strip by Using Its Curvatures

In this section we find energy of the strip by using classic energy formulaes $E_{T=} E_p + E_k$.

Now we repeat the Joachimsthal Theorems.

*Theorem 4.1.* (Terquem Theorem) Let $M_1$ and $M_2$ be the different surfaces in $E^3$ and $\alpha$ be a curve but not a planar curve and $\beta$ be a curve in $M_2$.

*i.* The points of the curves $\alpha$ ve $\beta$ corresponds to each other 1:1 on a plane $\varepsilon$ which rolls on the $M_1$ and $M_2$, such that the distance is constant between corresponding points.

*ii.* $(\alpha, M_1)$ is a curvature strip.

*iii.* $(\beta, M_2)$ is a curvature strip.

*Proof.* Claim: Two of the three lemmas gives third ([10]). It is obviously from the Phd. thesis by Keles.

By applying the similar way in proof of the Theorem 3.1 in [10] to the strip of the spherical helix strip, we give the following theorem.

*Theorem 4.2.* (Joachimsthal Theorem) Let $S^2$ be a sphere and and $M$ be a surface in $E^3$. Let the tangent planes of the surface $M$ that along the curve $\beta$ be the tangent planes of the sphere $S^2$ along the helix curve $\alpha$ at the same time. In this case, if we find the energy of the strip $(\beta, M)$, the curve $\beta$ is a helix, also a helix strip. If we find the energy of the curve $\alpha$ on the spherical helix strip ( $S^2$ , $M$ ), we can find the energy of the curve $\beta$ on ( $\beta$ , $M$ ) in type of the curvatures of the ( $\beta$ , $M$ ) and give a characterization.

*Proof:*

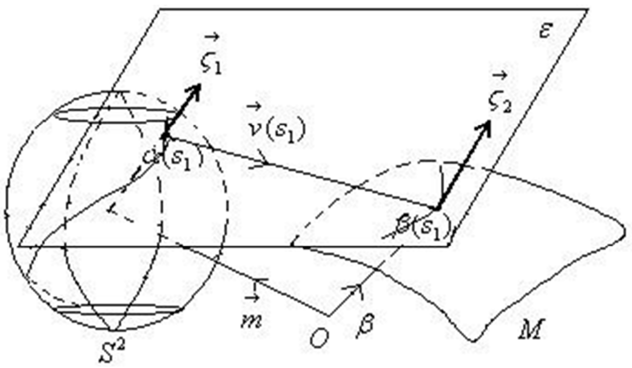


***Figure 1.*** *The spherical helix $S^2$ and the surface $M$.*

Now Keles's proof help us to obtain the energy of the strip.

If the curve $\alpha$ is a helix on $S^2$, then it provides $\frac{\kappa_1}{\tau_1}$ is constant. We have to show that $\beta$ is a helix strip on $M$, that is, $\frac{\kappa_2}{\tau_2}$ is constant.

By the Figure, we have

$$\beta(s_1) = \alpha(s_1) + \lambda(s_1)\vec{\nu}(s_1) \tag{2}$$

where

$$\alpha(s_1) = \vec{m} + r\zeta_1(s_1) \tag{3}$$

By differentiating both side of (3), we see that

$$\vec{\xi}_1 = \frac{d\alpha_1}{ds_1} = r\frac{d\zeta_1}{ds_1}.$$

By (1),

$$\vec{\xi}_1 = r(b_1\xi_1 - a_1\eta_1),$$

We obtain $a_1 = 0$ and $b_1 = 1$.

$r$ is the radius of the sphere. We denote $r = 1$. Since $\vec{m}$ is a position vector that goes to the center of the sphere, $\vec{m}$ is constant.

Since $\alpha_1 = 0$, $(\alpha, S^2)$ is a curvature strip. By the strips $(\alpha, S^2)$ and $(\beta, M)$ are curvature strips and by the Terquem Theorem, we see that $\lambda$ is non-zero constant. Let $\vec{\nu}(s_1)$ be a vector in $Sp\{\xi_1, \eta_1\}$, and let $\phi$ be the angle between $\vec{\xi}_1$ and $\vec{\nu}(s_1)$. Then we write

$$\vec{\nu}(s_1) = \cos\phi\,\vec{\xi}_1 + \sin\phi\,\vec{\eta} \tag{4}$$

By substituting (3) and (4) in (2), and differentiating both sides, we obtain (5).

$$\frac{d\beta}{ds_1} = \frac{d\vec{m}}{ds_1} + \frac{d\vec{\zeta}_1}{ds_1} + \frac{d\lambda}{ds_1}(\cos\phi\,\vec{\xi}_1 + \sin\vec{\eta}_1) + \lambda(s_1)\frac{d(\cos\phi\,\vec{\xi}_1 + \sin\phi\,\vec{\eta}_1)}{ds_1} \tag{5}$$

Since the vector $\vec{m}$ and $\lambda$ are constant, we obtain the following equation

$$\frac{d\beta}{ds_1} = \frac{d\vec{\zeta}_1}{ds_1} + \lambda(s_1)\frac{d(\cos\phi\,\vec{\xi}_1 + \sin\phi\,\vec{\eta}_1)}{ds_1}$$

or

$$\frac{d\beta}{ds_1} = \frac{d\vec{\zeta}_1}{ds_1} + \lambda(s_1)(-\frac{d\phi}{ds_1}\sin\phi\,\vec{\xi} + \cos\phi\frac{d\xi_1}{ds_1}) + \frac{d\phi}{ds_1}\cos\phi\,\vec{\eta}_1 + \sin\phi\frac{d\vec{\eta}_1}{ds_1}).$$

By (1), we obtain

$$\frac{d\beta}{ds_1} = \left[1 - \lambda(\frac{d\varphi}{ds_1} + c_1)\sin\phi\right]\vec{\xi}_1 + \lambda(\frac{d\phi}{ds_1} + c_1)\cos\phi\,\vec{\eta}_1 - \lambda\cos\phi\,\vec{\zeta}_1 \tag{6}$$

This $\frac{d\beta}{ds}$ is the velocity of the curve.

Since the spherical helix and the surface $M$ have the same tangent plane along the curves $\alpha$ and $\beta$, we can write

$$\left\langle \frac{d\beta}{ds_1}, \vec{\zeta}_1 \right\rangle = 0.$$

By substituting (6) at the last equation, we obtain $\cos\phi = 0$. By using that equation in (6), we have

$$\frac{d\beta}{ds_1} = (1 \pm \lambda c_1)\vec{\xi_1} \tag{7}$$

Since the same result is obtained by using other form of (7), we use the form $\frac{d\beta}{ds_1} = (1 - \lambda c_1)\vec{\xi_1}$ of (7) at the rest of our proof. By differentiating both sides of (7), we obtain

$$V_1 = \frac{d\beta}{ds_1} = (1 - \lambda c_1)\vec{\xi_1}(meter / s)$$

Now calculate the $E_k$ of the slant helix,
Special case:
Take $m = 1$ kg, so

$$E_k = \frac{1}{2} m.V^2 = \frac{1}{2}.1(kg).[(1-\lambda c_1)\xi_1(\vec{m}/s)]^2 = \frac{\xi_1^2}{2}(1 - 2\lambda c_1 - c_1^2)(kgm^2 / s^2).$$

Now calculate the $E_p$ of the slant helix,
Take $m = 1$ kg and $h = 2r = 2.1 = 2meters$

$$E_p = m.g.h = 1(kg).9{,}8(m/s^2).2(m) = 19{,}6(kgm^2/s^2)$$

So the total energy is

$$E_{Total=} E_p + E_k = 19{,}6(kgm^2/s^2) + \frac{\xi_1^2}{2}(1 - 2\lambda c_1 - c_1^2)(kgm^2/s^2)$$

$$E_{Total=} \left(19{,}6 + \frac{\xi_1^2}{2}(1 - 2\lambda c_1 - c_1^2)\right)(kgm^2/s^2)$$

We will use this equations for finding energy of the strip on Joachimsthal theorem by using its curvatures.

## Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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