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Fixed Point Result on Generalized Cone b-Metric Spaces

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Abstract: The purpose of this paper is to prove a fixed point result on contraction mapping in generalized cone b-metric space (in short GCbMS) as a generalization of cone metric space, cone b metric space and rectangular metric space. The conception of generalized metric space is a generalization of that of classical metric space. Several authors have proved fixed point theorems of contractive mappings on generalized metric spaces, which also generalized some corresponding fixed point results in classical metric spaces. In present paper, we prove a result that is extension of the Kannan fixed point theorem proved by Reny George *et al*. Our result is extend and unify several well known results in the literature available for cone and cone-b metric space.

Keywords: Metric Space, b-metric Space, Cone Metric Space, Cone b-metric Space, Rectangular Metric Space

1. Introduction

Fixed point theorems have wide applications in different fields of mathematics. Due to which, existence as well as uniqueness of fixed points and common fixed points has turn into a subject of great importance. This is due to verify Banach Contraction Principle in different directions. In the recent four to five decades many authors generalized the Banach contraction Principle by moderating the triangular inequality of a metric space. generalized metric space [2, 7, 9-10, 16, 25], cone metric space [11], b metric space [4, 5, 6, 8 references therein], cone b metric space [11, 12, 13, 16-22], rectangular metric space [21], cone rectangular metric space [14, 19, 20], are some of the generalizations of metric space introduced by different authors in past few decades. Analogue Banach contraction principle, Kannan contraction principle, Ciric contraction principle and lots of the existing fixed-point theorems for various generalized contractions were proved in these generalized spaces.

Most of the generalization of metric space are Hausdorff topology but we can also find generalization of metric space which are not necessarily Hausdorff topology [15, 21, 24, 25]. Tarskian mathematician used non Hausdorff topology to programming language semantics used in computer science. Here, we prove a fixed point theorem for contraction mapping in generalized cone b-metric space (in short GCbMS) as a generalization of cone metric space, cone b metric space and rectangular metric space which is the extension of Kannan fixed point theorem proved by Reny George *et al*. [17].

2. Preliminaries

(1) Let E be a real Banach spaces and P $\subset E$. P is called a cone iff
  (i) P is closed and non-empty and P$\neq \{\theta\}$
  (ii) $ax + by \in P \; \forall \, x, y \in P$ and a, b are non negative real
  (iii) if $x \in P; -x \in P \Rightarrow x = 0$ i.e. $P \cap -P = \{\theta\}$

Given a cone $P \subset$ E we define a partial ordering $\preccurlyeq$ with respect to P as
  (i) $x \preccurlyeq y$ if and only if $y - x \in P$
  (ii) $x \preccurlyeq y$ & $x \neq y$, then we may write $x < y$
  (iii) $x \ll y \Rightarrow y - x \in int\, P$ and
  (iv) if $int\, P \neq \emptyset$, cone P is called solid cone.
  (v) The cone P is said to be Normal if there exist a number $k > 0 \; \forall \, x, y \in E$

$$0 \leq x \leq y \Rightarrow \|x\| \preccurlyeq k\|y\|$$

The least positive number satisfying above is called the normal constant of P.

(vi) The cone is said to be regular if every increasing sequence $\{x_n\}$ which is bounded from above is convergent i.e. if $\{x_n\}$ is a sequence $x_1 \leq x_2 \leq x_3 \leq \cdots x_n \leq \cdots \leq y$ *some* $y \in E \exists x \in E$ such that $\lim_{n\to\infty} x_n = 0$.

Equivalently the cone is said to be regular if every increasing sequence $\{x_n\}$ which is bounded from below is convergent i.e. if $\{x_n\}$ is a sequence $x_1 \geq x_2 \geq x_3 \geq \cdots x_n \geq \cdots \geq y$ *some* $y \in E \exists x \in E$, such that $\lim_{n\to\infty} x_n = 0$.

Note: A regular cone is a normal cone.

(2) Definition: Let E be a topological vector space over the field $F = R$ *or* C and let $P \subseteq E$ be a closed set with following axioms:

(C1) $P \neq \emptyset, \{0\}$

(C2) $x, y \in P$ & $\delta \geq 0 \Rightarrow x + y, \delta x \in P$

(C3) $P \cap -P = \{0\}$,

Then P is called a cone. If, in addition, $int\ P \neq \emptyset$, we say that P is a solid cone.

(3) Definition: Let X be a non-empty set and $\varphi: X \to X$ is a mapping such that $\forall x, y, z \in X$ satisfies

(CMS1) $0 \preccurlyeq \varphi(x, y)\ \forall x, y$ & $x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CMS2) $\varphi(x, y) = \varphi(y, x)\ \forall x, y$ & $x \neq y$

(CMS3) $\varphi(x, y) \preccurlyeq \varphi(x, z) + \varphi(z, y)$

Then φ is called a cone metric on X and (X, φ) is called cone metric space.

(4) Definition: The pair (X, d) is called b-metric space with coefficient $s \geq 1$ such that X is a non empty set and a mapping $d: X \times X \to R^+$ satisfies the following axioms for all $x, y, z \in X$

(bMS_1) $d(x, y) = 0$ if and only if $x = y$

(bMS_2) $d(x, y) = d(y, x)$

(bMS_3) $d(x, y) \leq s\{d(x, z) + d(z, y)\}$

(5) Example: Let $X = R$ be the set of real number and $d(x, y) = |x - y|$ a usual metric, then $\varphi(x, y) = |x - y|$ is a b-metric space for $k = 2$ but not for R.

(6) Example: The space l_p $(o < p < 1)$

$$l_p = \{ (x_n \in R: \textstyle\sum_{n=1}^{n} |x_n|^p < \infty\}$$

Together with the mapping $d: l_p \times l_p \to R$

$$d(x, y) = (\textstyle\sum_{n=1}^{n} |x_n - y_n|^p)^{\frac{1}{p}}$$

is a b-metric spaces.

(7) Definition: Let X be a non-empty set, $s \geq 1$ is a real and $\varphi: X \to X$ is a mapping such that $\forall x, y, z \in X$ satisfies

(CBM1) $0 \preccurlyeq \varphi(x, y)\ \forall x, y$ & $x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CBM2) $\varphi(x, y) = \varphi(y, x)\ \forall x, y$ & $x \neq y$

(CBM3) $\varphi(x, y) \preccurlyeq s[\varphi(x, z) + \varphi(z, y)]$

Then φ is called a Cone b-Metric on X and (X, φ) is called Cone b- Metric Space.

(8) Example: Let (X, φ) be a metric space and $\varphi(x, y) = (d(x, y))^p$ *where* $p > 1$, Then (X, φ) is a Cone b-metric space for $s = 2^{p-1}$.

Proof: (i) if $x = y$ *then* $\varphi(x, x) = (d(x, x))^p = 0$

$$(ii)\varphi(x, y) = (d(x, y))^p = (d(y, x))^p = \varphi(y, x)$$

(iii) if $0 < p < 1$ then by convexity of the function $f(x) = x^p \Rightarrow (\frac{a+b}{2})^p \leq \frac{1}{2}(a^p + b^p) \Rightarrow (a + b)^p = 2^{p-1}(a^p + b^p)$. So for $x, y, z \in X$, we have

$$\varphi(x, y) = (d(x, y))^p \leq (d(x, z) + d(z, y))^p \leq 2^{p-1}(d(x, z))^p + (d(z, y))^p \leq 2^{p-1}\{\varphi(x, z) + \varphi(z, y)\}$$

Hence (X, φ) is a Cone b-metric space.

(9) Definition: Let X be a non-empty set and $\varphi: X \to X$ is a mapping such that $\forall x, y, z, a, b \in X$ satisfies

(CRM1) $0 \preccurlyeq \varphi(x, y)\ \forall x, y$ & $x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(CRM2) $\varphi(x, y) = \varphi(y, x)\ \forall x, y$ & $x \neq y$

(CRM3) $\varphi(x, y) \preccurlyeq \varphi(x, a) + \varphi(a, b) + \varphi(b, y)$

Then φ is called a Cone Rectangular Metric on X and (X, φ) is called Cone Rectangular Metric Space.

(10) Definition: Let X be a non empty set, $s \geq 1$ is a real and $\varphi: X \to X$ is a mapping such that $\forall x, y, z, a, b \in X$ satisfies

(GCRMS1) $0 \preccurlyeq \varphi(x, y)\ \forall x, y$ & $x \neq y$ and $\varphi(x, y) = 0$ if and only if $x = y$

(GCRMS2) $\varphi(x, y) = \varphi(y, x)\ \forall x, y$ & $x \neq y$

(GCRMS3) $\varphi(x, y) \preccurlyeq s[\varphi(x, a) + \varphi(a, b) + \varphi(b, y)]$

Then φ is called a Generalized Cone Rectangular Metric on X and (X, φ) is called Generalized Cone.

(11) Example: Let $E = R^2, P = \{(x, y) \in E: x, y \geq 0\}, X = R = A \cup B,, d: X \times X \to E, \forall x, y \in R$

$d(x, y) = 0$ *if* $x = y$ *and* $d(x, y) = d(y, x)$ such that

$$d(x, y) = \begin{cases} (12,12) \text{ for } x = 3, y = 2 \\ (2,2) \text{ for } x \in \{2,3\} \text{ and } y = 4 \\ (4,4) \text{ for } x \in \{2,3,4\} \text{ and } y = 5 \\ (2,3) \text{ for } x \text{ or } y \notin \{2,3,4,5\} \text{ and } x \neq y \end{cases}$$

T hen (X, d) is a GCbMS but not a CRMS as we have

$$d(3,2) = (12,12) \succ d(3,4) + d(4,5) + d(5,2) = (2,2) + (4,4) + (4,4) = (10,10)$$

MAIN RESULTS: Reny George et. al. [17] proved Banach contraction principle and Kannan contraction principle. We extended their results in the present paper.

Theorem 3: Let (X, φ) be a complete generalized cone b-metric space with constant coefficient $s \geq 1$ and let $\alpha_i \geq 0 (i = 1,2,3,4)$, P is a solid cone and $T: X \times X \to X$ be a mapping satisfying;

$$\varphi(Tx,Ty) \preccurlyeq \alpha_1\varphi(x,y) + \alpha_2\frac{[\varphi(x,Tx)+\varphi(y,Ty)]}{2} + \alpha_3\frac{[\varphi(x,y)+\varphi(Tx,Ty)]}{2} + \alpha_4\frac{[\varphi(x,Ty)+\varphi(y,Tx)]}{2s} \tag{1}$$

for all $x, y \in X$ where $s\alpha_1 + \frac{(1+s)}{2}(\alpha_2+\alpha_3+\alpha_4) \leq 1$. Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ be a arbitrary point. Define a sequence $\{x_n\} \in X$ such that $Tx_n = x_{n+1} (n \geq 0)$. Now we are to show that sequence $\{x_n\}$ is a Cauchy sequence. If for any $n; x_n = x_{n+1}$, then x_n is a unique fixed point for mapping T. Therefore, there is no need to go further. Otherwise $x_{n+1} \neq x_n \; \forall\, n \geq 1$; setting $\varphi_n = \varphi(x_n, x_{n+1})$. Thus inserting$x_{n-1}$; x_n for x; y respectively in inequality (3.1), we have

$$\varphi(x_n, x_{n+1}) = \varphi(Tx_{n-1}, Tx_n) \preccurlyeq$$

$$\alpha_1\varphi(x_{n-1},x_n) + \alpha_2\frac{[\varphi(x_{n-1},Tx_{n-1})+\varphi(x_n,Tx_n)]}{2} + \alpha_3\frac{[\varphi(x_{n-1},x_n)+\varphi(Tx_{n-1},Tx_n)]}{2} + \alpha_4\frac{[\varphi(x_{n-1},Tx_n)+\varphi(x_n,Tx_{n-1})]}{2s}$$

$$\varphi(x_n,x_{n+1}) \preccurlyeq \alpha_1\varphi(x_{n-1},x_n) + \alpha_2\frac{[\varphi(x_{n-1},x_n)+\varphi(x_n,x_{n+1})]}{2} + \alpha_3\frac{[\varphi(x_{n-1},x_n)+\varphi(x_n,x_{n+1})]}{2} + \alpha_4\frac{[\varphi(x_{n-1},x_{n+1})+\varphi(x_n,x_n)]}{2s}$$

$$\varphi_n \preccurlyeq \alpha_1\varphi_{n-1} + \alpha_2\frac{[\varphi_{n-1}+\varphi_n]}{2} + \alpha_3\frac{[\varphi_{n-1}+\varphi_n]}{2} + \alpha_4\frac{s[\varphi_{n-1}+\varphi_n]}{2s}$$

$$\left(1-\frac{\alpha_2}{2}-\frac{\alpha_3}{2}-\frac{\alpha_4}{2}\right)\varphi_n \preccurlyeq \left(\alpha_1+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}\right)\varphi_{n-1}$$

$$\varphi_n \preccurlyeq \frac{\left(\alpha_1+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}\right)}{\left(1-\frac{\alpha_2}{2}-\frac{\alpha_3}{2}-\frac{\alpha_4}{2}\right)}\varphi_{n-1}$$

$$\varphi_n \preccurlyeq \gamma\varphi_{n-1}$$

Where

$$\gamma = \frac{\left(\alpha_1+\frac{\alpha_2}{2}+\frac{\alpha_3}{2}+\frac{\alpha_4}{2}\right)}{\left(1-\frac{\alpha_2}{2}-\frac{\alpha_3}{2}-\frac{\alpha_4}{2}\right)} \leq \frac{1}{s} \Rightarrow s\alpha_1 + \frac{(1+s)}{2}(\alpha_2+\alpha_3+\alpha_4) \leq 1.$$

Repeating iteration n times; we have

$$\varphi_n \preccurlyeq \gamma^n\,\varphi_0 \tag{2}$$

Assume;$x_0$is not a periodic point of φ. Indeed, if$x_n = x_0$ then; from (3.2), for any $n \geq 2$; we conclude that

$$\varphi_0 = \varphi(x_0,x_1) = \varphi(x_0,Tx_0) = \varphi(x_n,Tx_n) = \varphi(x_n,x_{n+1}) = \varphi_n \leq \gamma^n\,\varphi_0$$

Since $\gamma \in [0,1)$; we obtain $-\varphi_0 \in P \Rightarrow \varphi_0 = 0 \Rightarrow x_0 = x_1$ i.e. x_0 is a fixed point for T. Therefore we assume that $x_n \neq x_m \; \forall\, n, m \in N \; and \; n \neq m$, again letting $\varphi_n^* = \varphi(x_n, x_{n+2})$ and using (3.1), we observe that

$$\varphi_n^* = \varphi(x_n,x_{n+2}) = \varphi(Tx_{n-1},Tx_{n+1}) \preccurlyeq \gamma\varphi(x_{n-1},x_{n+1}) \preccurlyeq \gamma\varphi_{n-1}^*$$

Repeating iteration n times; we have $\varphi_n^* \preccurlyeq \gamma^n\varphi_0^*$

Now for any sequence $\{x_n\}$, two cases may be arising for $\varphi(x_n, x_{n+p})$

(1) $p = 2m+1$ i.e. p is odd. Letting $n \geq 2$ and $m > n$; we have

$$\varphi(x_n,x_{n+2m+1}) \preccurlyeq s\,[\varphi(x_n,x_{n+1}) + \varphi(x_{n+1},x_{n+2}) + \varphi(x_{n+2},x_{n+2m+1})]$$

$$\varphi(x_n,x_{n+2m+1}) \leq s(\varphi_n+\varphi_{n+1})$$

$$+s^2\,[\varphi(x_{n+2},x_{n+3}) + \varphi(x_{n+3},x_{n+4}) + \varphi(x_{n+4},x_{n+2m+1})]$$

$$\varphi(x_n,x_{n+2m+1}) \preccurlyeq s(\varphi_n+\varphi_{n+1}) + s^2[(\varphi_{n+2}+\varphi_{n+3})]$$

$$+s^2\,[\varphi(x_{n+4},x_{n+5}) + \varphi(x_{n+5},x_{n+2m+1})]$$

$$\varphi(x_n,x_{n+2m+1}) \preccurlyeq s(\varphi_n+\varphi_{n+1}) + s^2[(\varphi_{n+2}+\varphi_{n+3})] + s^3[(\varphi_{n+4}+\varphi_{n+5})] + \cdots$$

$$+s^{m-1}[\varphi(\varphi_{n+2m-2}+\varphi_{n+2m-1})] + s^m\varphi_{n+2m}$$

$$\varphi(x_n,x_{n+2m+1}) \preccurlyeq s(\gamma^n\varphi_0 + \gamma^{n+1}\varphi_0) + s^2(\gamma^{n+2}\varphi_0 + \gamma^{n+3}\varphi_0)$$

$$+s^3(\gamma^{n+4}\varphi_0 + \gamma^{n+5}\varphi_0) + \cdots + s^{m-1}(\gamma^{n+2m-2}\varphi_0 + \gamma^{n+2m-1}\varphi_0) + s^m\gamma^{n+2m}\varphi_0$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq (s\gamma^n + s^2\gamma^{n+2} + \cdots)\varphi_0 + (s\gamma^{n+1} + s^2\gamma^{n+3} + \cdots)\varphi_0$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n(1 + s\gamma^2 + s^2\gamma^4 + \cdots)\varphi_0 + s\gamma^{n+1}(1 + s\gamma^2 + s^2\gamma^4 + \cdots)\varphi_0$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0$$

Let $0 \ll c$ (given). Now choose a natural N_π such that $c + N_\pi(0) \subseteq P$, here $N_\pi = \{y \in E, \|y\| < \pi\}$. Now choose another natural $N_{\pi 1}$ such that $s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \in N_\pi(0)\ \forall\, n \in N_{\pi 1}$, so $s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \ll c\ \forall\, n \in N_{\pi 1}$, implies $\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \ll c\ \forall\, n \in N_{\pi 1}$.

(2) $p = 2m$ i.e., p is even. Letting $n \geq 2$ and $m > n$; we have

$$\varphi(x_n, x_{n+2m}) \preccurlyeq s\,[\varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, x_{n+2}) + \varphi(x_{n+2}, x_{n+2m})]$$

$$\varphi(x_n, x_{n+2m+1}) \leq s(\varphi_n + \varphi_{n+1})$$

$$+s^2\,[\varphi(x_{n+2}, x_{n+3}) + \varphi(x_{n+3}, x_{n+4}) + \varphi(x_{n+4}, x_{n+2m})$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s(\varphi_n + \varphi_{n+1}) + s^2[(\varphi_{n+2} + \varphi_{n+3})]$$

$$+s^2\,[\varphi(x_{n+4}, x_{n+5}) + \varphi(x_{n+5}, x_{n+2m+1})]$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s[(\varphi_n + \varphi_{n+1})] + s^2[(\varphi_{n+2} + \varphi_{n+3})]$$

$$+s^3[(\varphi_{n+4} + \varphi_{n+5})] + \cdots + s^{m-1}(\varphi_{n+2m-2} + \varphi_{n+2m)}$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s(\gamma^n\varphi_0 + \gamma^{n+1}\varphi_0) + s^2(\gamma^{n+2}\varphi_0 + \gamma^{n+3}\varphi_0)$$

$$+s^3(\gamma^{n+4}\varphi_0 + \gamma^{n+5}\varphi_0) + \cdots + s^{m-1}(\gamma^{n+2m-2}\varphi_0 + \gamma^{n+2m-1}\varphi_0) + s^{m-1}\gamma^{n+2m-2}\varphi_0^*$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq (s\gamma^n + s^2\gamma^{n+2} + \cdots)\varphi_0 + (s\gamma^{n+1} + s^2\gamma^{n+3} + \cdots)\varphi_0$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n(1 + s\gamma^2 + s^2\gamma^4 + \cdots)\varphi_0$$

$$+s\gamma^{n+1}(1 + s\gamma^2 + s^2\gamma^4 + \cdots)\varphi_0 + s^{m-1}\gamma^{n+2m-2}\varphi_0^*$$

$$\varphi(x_n, x_{n+2m+1}) < s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 + (s\gamma)^{2m}\gamma^{n-2}\varphi_0^*$$

$$\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n \tfrac{1+\gamma}{1-s\gamma^2}\varphi_0 + \gamma^{n-2}\varphi_0^* \text{ since } \gamma < \tfrac{1}{s}$$

Let $0 \ll c$ (given). Now choose a natural $N_{\pi 2}$ such that$c + N_{\pi 2}(0) \subseteq P$, here$N_{\pi 2} = \{y \in E, \|y\| < \pi\}$. Now choose another natural $N_{\pi 3}$ such that$s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \in N_\pi(0)\ \forall\, n \in N_{\pi 2}$, so
$s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \ll c\ \forall\, n \in N_{\pi 2}$, implies $\varphi(x_n, x_{n+2m+1}) \preccurlyeq s\gamma^n \frac{1+\gamma}{1-s\gamma^2}\varphi_0 \ll c\ \forall\, n \in N_{\pi 2}$. Let $N_\pi = Max\{N_{\pi 1}; N_{\pi 2}\}\ \forall\, n \geq N_\pi \Rightarrow\ \log_{n\to\infty}\varphi(x_n, x_{n+p}) \ll c$, implies $\{x_n\}$ is a Cauchy sequence.

Therefore; due to completeness of metric spaces (X, φ), there exist an element $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Now we shall show that a is a fixed point for$T$. For that assume$n \in N$, we have

$$\varphi(a, Ta) \preccurlyeq s[\varphi(a, x_n) + \varphi(x_n, x_{n+1}) + \varphi(x_{n+1}, Ta)$$

$$\varphi(a, Ta) \preccurlyeq s[\varphi(a, x_n) + \varphi_n + \varphi(Tx_n, Ta)]$$

$$\varphi(\alpha, T\alpha) \preccurlyeq s[\varphi(\alpha, x_n) + \gamma^n\varphi_0 + \gamma\varphi(x_n, \alpha)]$$

$$\varphi(a, Ta) \preccurlyeq s[\varphi(x_n, a) + \gamma^n\varphi_0 + \gamma\varphi(x_n, a)] \text{ from (3.2)}$$

$$\varphi(a, Ta) \leq s[(1+\gamma)\varphi(x_n, a) + \gamma^n\varphi_0)$$

Now choose a natural $N_{\pi 3}; N_{\pi 4}$ such that $\varphi(x_n, a) \ll \frac{c}{2s(1+\gamma)}\ \forall\, n \in N_{\pi 3}$ and $\gamma^n\varphi_0 \ll \frac{c}{2s}\ \forall\, n \in N_{\pi 4}$. Let $N_\pi = Max\{N_{\pi 1}; N_{\pi 2}\}\ \forall\, n \geq N_\pi$so that $\varphi(a, Ta) \preccurlyeq c\ \Rightarrow \varphi(a, Ta) = 0 \Rightarrow a = Ta$ implies a is a fixed point.

For uniqueness; assume b is another fixed point of T. Therefore from (3.1), we have

$$\varphi(a,b) = \varphi(Ta,Tb) \preccurlyeq \alpha_1\varphi(a,b) + \alpha_2\frac{[\varphi(a,Ta)+\varphi(b,Tb)]}{2}$$

$$+\alpha_3\frac{[\varphi(a,b)+\varphi(Ta,Tb)]}{2} + \alpha_4\frac{[\varphi(a,Tb)+\varphi(b,Ta)]}{2s}$$

$$\varphi(a,b) \preccurlyeq \alpha_1\varphi(a,b) + \alpha_2\frac{[\varphi(a,a)+\varphi(b,b)]}{2} + \alpha_3\frac{[\varphi(a,b)+\varphi(a,b)]}{2} + \alpha_4\frac{[\varphi(a,b)+\varphi(a,b)]}{2s}$$

$$\varphi(a,b) \preccurlyeq \alpha_1\varphi(a,b) + \alpha_3\varphi(a,b) + \frac{\alpha_4}{s}\varphi(a,b)$$

$$\varphi(a,b) \preccurlyeq (\alpha_1 + \alpha_3 + \frac{\alpha_4}{s})\,\varphi(a,b) \Rightarrow \varphi(a,b) = 0, since\ (\alpha_1 + \alpha_3 + \frac{\alpha_4}{s}) < 1.$$

This implies that a and b are not different points but are same. Hence ‘a’ is a unique fixed point of T.

This completes the proof.

3. Conclusion

In present paper, the concept of cone metric spaces and generalized cone b-metric space on fixed points with an example illustrated. Also proved a result on contractive mapping.

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