Pure and Applied Mathematics Journal
2021; 10(2): 49-61
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20211002.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Unstructured and Semi-structured Complex Numbers: A Solution to Division by Zero

Peter Jean-Paul[1, *], Shanaz Wahid[1, 2]

[1]School of Engineering, Computer and Mathematical Sciences, Design and Creative Technologies, Auckland University of Technology, Auckland, New Zealand

[2]Mathematics & Statistics, Faculty of Science and Technology, University of the West Indies, St. Augustine, Trinidad and Tobago

Email address:

peter.jeanpaulnz@gmail.com (P. Jean-Paul), shanaz.wahid@sta.uwi.edu (S. Wahid)

[*]Corresponding author

To cite this article:

Peter Jean-Paul, Shanaz Wahid. Unstructured and Semi-structured Complex Numbers: A Solution to Division by Zero. *Pure and Applied Mathematics Journal*. Vol. 10, No. 2, 2021, pp. 49-61. doi: 10.11648/j.pamj.20211002.12

Received: April 16, 2021; **Accepted**: April 30, 2021; **Published**: May 8, 2021

Abstract: Most mathematician, have accepted that a constant divided by zero is undefined. However, accepting this situation is an unsatisfactory solution to the problem as division by zero has arisen frequently enough in mathematics and science to warrant some serious consideration. The aim of this paper was to propose and prove the existence of a new number set in which division by zero is well defined. To do this, the paper first uses set theory to develop the idea of unstructured numbers and uses this new number to create a new number set called "Semi-structured Complex Number set" ($\acute{S}$). It was then shown that a semi-structured complex number is a three-dimensional number which can be represented in the *xyz*-space with the x-axis being the real axis, the y-axis the imaginary axis and the z-axis the unstructured axis. A unit of rotation p was defined that enabled rotation of a point along the *xy*-, *xz*- and *yz*- planes. The field axioms were then used to show that the set is a "complete ordered field" and hence prove its existence. Examples of how these semi-structured complex numbers are used algebraically are provided. The successful development of this proposed number set has implications not just in the field of mathematics but in other areas of science where division by zero is essential.

Keywords: Unstructured Numbers, Semi-structured Complex Number, Zero

1. Introduction

1.1. The Problem

The development of the current number system that is used to today went through several evolutionary stages that arise out of a necessity to find solutions to several mathematical problems. Initially, the set of whole numbers $\{1,2,3,4,\cdots\}$ was developed to provide simple arithmetic. But this set was soon considered not sufficient and led to the development of the natural number set $\{0,1,2,3,4,\cdots\}$. The set of natural number eventually led to the set of integers and the set of integers to the set of rational numbers. However rational numbers were not sufficient explain numbers that have no exact factional representation such as "π" or "e". Hence, the real number set was developed.

Nevertheless, even real numbers became inadequate when faced with the problem of determining "the result of $\sqrt{-1}$". Therefore, over a period of centuries the complex numbers were developed. With the set of complex numbers, $\sqrt{-1} = i$. Here "i" was given the misnomer "imaginary number". The complex numbers were given the form $a + bi$, where "a" and "b" are real numbers. This new number system permitted polynomial equations with $\sqrt{-1}$ as the solution to be solved. In addition, the unit "i" was considered the unit of rotation that enabled a point to be rotated along the complex plane i.e., the xy-plane as shown in Figure 1.

Nevertheless, there still exists another unsolved mathematical problem; determining the solution to $\frac{a}{0}$, where "a" belongs to the set of real numbers. The current number systems developed are inadequate in giving a viable solution to this problem. Therefore, the most logical step would be to develop a new number set that can provide a feasible solution to this problem. Not only must the new number set be developed but it must also have a feasible geometric solution as well as satisfy the field axioms to prove its existence.

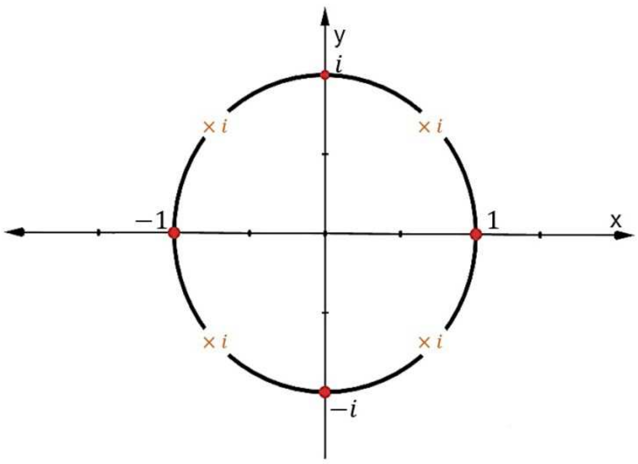

Figure 1. *Rotation about the real and complex axis using unit i.*

Axioms are rules or properties that are accepted as true without proof. Axioms are important because they help create a framework from which mathematical theorems can be developed and checked. It is important to note that axioms cannot contradict each other. A field on the other hand is a set of numbers (or other objects) that are "closed" under the operations of addition, subtraction, multiplication, and division. "Closed" means that if any of these arithmetic operations are done on two or more of the numbers in the field it will lead to another number in the field. Examples of fields include: Rational numbers, real number, and complex numbers. Fields are important because they are used to solve problems in polynomial equations, algebraic number theory, and algebraic geometry.

Field axioms are therefore properties of a field that are accepted without proof. Field axioms help create a framework from which mathematical theorems about fields can be develop and checked. They also help prove the existence of a number set as any number set that is a field must satisfy these axioms. There are 11 field axioms that must be satisfied for a number set to be considered a field [1]. These are given in appendix 1. The problem therefore becomes an issue of:

Developing a number set in which the solution to $\frac{a}{0}$, (where "a" belongs to the set of real numbers) is well defined, has a geometric interpretation and satisfies the field axioms.

1.2. Previous Attempts That Have Been Made at a Solution

For 1200 year several mathematicians have approached the problem of division by zero. Some of the earliest examples are given in Table 1.

Table 1. *Some early notable examples of mathematicians approaching the topic of division by zero [2].*

Attempt	Date
1) Brahamagupta claimed that $\frac{0}{0} = 0$, while $\frac{n}{0}$(with $n \neq 0$) is a fraction with zero as denominator.	628 A.D.
2) Mahāvīra generalized the former result of Brahamagupta claiming that $\frac{n}{0} = 0$ where n is any number. Mahavira stated that dividing by zero is equivalent to not dividing.	830 A.D.
3) Bhāskara II supposed that $\frac{n}{0} = \infty$	1150 A.D.
4) George Berkeley's criticism of infinitesimal calculus in 1734 in The Analyst ("ghosts of departed quantities") noted the absurdity of dividing by zero	1734
5) Euler's thesis showed that $n = \infty \times 0$ where n is every number	1770
6) Several structures created such as 19th century Reiman Sphere in which division by zero is allowed. However, Reiman sphere structure does not satisfy the existence axioms	1800's
7) Several fallacies were made public and listed as being reasons for not dividing by zero. For example, if $0 \times 1 = 0$ and $0 \times 2 = 0$ then f $0 \times 1 = 0 \times 2$. This implies f $1 = 2$ which is a fallacy	1900's

Within the last century there has been three more recent notable examples of persons who have attempted to solve division by zero. In 1997, wheel theory [3] was developed to deal with the problem of division by zero. In their paper the authors suggested adding two extra elements to the set of real numbers. These two elements are $\frac{1}{0} = \infty$ and $\frac{0}{0} = \perp$. However, the criticism with wheel theory is that these extra elements would change the real number set from a field to a field with two extra elements.

In 2006 James Anderson a retired member in the School of Systems Engineering at the University of Reading, England claimed to have solved the problem of division by zero by developing "trans-real" numbers [4]. However, Dr Anderson was criticized for his work and it was soon realized that that his ideas are just a variation of the standard IEEE 754 concept of "NaN" (Not a Number), which has been used on computers in floating point arithmetic for many years.

Other notable examples of attempts to solve division by zero include hyper real numbers and bottom type logic. However, none of these attempts satisfy the field axioms and lead to extra arithmetic that does not fit well within the current everyday mathematics.

1.3. Research Gap and Major Contributions

It is clear from the literature that three major problems exist:

1. There is no clear consensus of what division by zero means. Because there is no clear definition of what division by zero means there is no clear application for such an arithmetic operation.
2. Secondly, no one has attempted to show the existence of a number set that is comprehensive enough to account for division by zero yet simple enough to be used in everyday mathematics. that can account for this phenomenon. However, this number system must be general enough to be applied to various areas of science and mathematics.

Given these issues and potential implication of developing a division by zero number set, the purpose of this paper was to:

Develop a number set in which the solution to $\frac{a}{0}$, (where

"a" belongs to the set of real numbers) is well defined, has a geometric interpretation and satisfies the field axioms.

In the process of achieving this purpose the following major contributions are made by this paper:

1. Novel number type called an Unstructured number was created: A new number type called an "unstructured number' $\dot{s}$ such that represents the number of empty sets in a real number "s".
2. Novel number set called a "Semi-structured Complex Number set $\dot{S}$" was defined: The elements of $\dot{S}$ are 3-dimensional numbers of the form: $a + bi + sp$, where a, b, s are real numbers; "i" is the complex unit of rotation about the complex plane (xy-plane); "p" is the unstructured unit of rotation about the unstructured plane (xz-plane).
3. Geometric interpretation of semi-structured numbers is given: The geometric interpretation of "p" was given. The number $a + bi + sp$ can be represented as an ordered triple (a, b, s). This ordered triple is considered a point in a 3D space with an x-axis consisting of real numbers and a y-axis consisting of imaginary numbers and a z-axis consisting of unstructured numbers. In addition, the unstructured unit of rotation number p was interpreted as a counter-clockwise rotation about the xz-plane.
4. Field Axioms Proof: The field axioms were used to show that the Semi-structured Complex number set is a complete order field. This satisfies conditions to prove that the set does exist.

The rest of this paper is devoted to providing a detailed explanation of how the objective of this paper was achieved and how this in turn gave rise to the development of the major contributions outlined in this paper.

2. A new Number Set: Developing the Semi-structured Complex Numbers

2.1. Structured and Unstructured Mathematics

In the early 20th century mathematicians Ernst Zermelo and Abraham Fraenkel, proposed an axiomatic system in order to formulate a theory of sets free of paradoxes such as Russell's paradox. In their theory, called Zermelo–Fraenkel (ZF) set theory [5], natural numbers were defined recursively, letting $0 = \{\}$ (the empty set) and $n + 1 = n \cup \{n\}$ for every successive "n". In this way $n = \{0,1,\cdots,n-1\}$ for each natural number n. This definition has the property that n is a set containing n elements. In the past researchers that have attempted to tackle division by zero have only considered the numbers as algebraic elements. That is the structure of the numbers as defined by Zermelo–Fraenkel (ZF) set theory was not given consideration. This is the major weakness in past literature and this has become problematic in dealing with division by zero. The first few numbers in the set of natural numbers as defined by Zermelo–Fraenkel (ZF) set theory is shown in Table 2. Each number is defined in terms of the empty set $\emptyset$.

Table 2. *Definition of the numbers in the natural number set.*

Definition of each number in set notation	Translation
$0 = \emptyset$	$0 = 0$
$1 = \{\emptyset\}$	$1 = \{0\}$
$2 = \{\emptyset, \{\emptyset\}\}$	$2 = \{0, 1\}$
$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$	$3 = \{0,1,2\}$
$4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$	$4 = \{0,1,2,3\}$
$\vdots$	$\vdots$

For the purposes of this paper the set of natural numbers as defined by Zermelo–Fraenkel (ZF) set theory is called the "*structured number set*". The term "*structured numbers*" is used because as shown in Table 2 the empty sets $\emptyset$ for each natural number is arranged in a very specific way (with the use of curl brackets) to define each number.

This paper also termed the algebra performed on these numbers with the normal arithmetic operations $(-,+,\div,\times)$ as "*structured algebra*". With "*structured algebra*" division by zero is undefined and multiplication by zero is redundant. That is, with "*structured algebra*" the discrepancies are given in Table 3.

Table 3. *Discrepancies that occur with structured algebra.*

Discrepancies	Translation
$a \times 0 = 0$	Any number multiplied by zero leads to the same result, zero. This is an issue since $a \times 0 = b \times 0$ implies $a = b$
$\frac{1}{0} = undefined$	Division by zero is undefined
'a' is any real number	

At this point it is reasoned that if structured numbers and structured algebra has led to the discrepancies shown in Table 3, then a different perspective is required to deal with division and multiplication by zero. Suppose the number of empty sets in each number defined in Table 2 is unpacked and counted as shown in Table 4.

Table 4. *Unstructured (unpacked) numbers.*

Structured definition of each number in the natural number set	Unstructured (unpacked) number set	Number of empty sets (zeros) in each unstructured number
$0 = \emptyset$	$0 = \emptyset$	1
$1 = \{\emptyset\}$	$\dot{1} = \{\emptyset\}$	2
$2 = \{\emptyset, \{\emptyset\}\}$	$\dot{2} = \{\emptyset, \emptyset\}$	3
$3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$	$\dot{3} = \{\emptyset, \emptyset, \emptyset, \emptyset\}$	5
$4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$	$\dot{4} = \{\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset\}$	9
$\vdots$	$\vdots$	$\vdots$

The unstructured (unpacked) numbers shown in column 2 of Table 4 were created by simply removing the curly brackets from the structured numbers shown in column 1 of Table 4. In each unpacked number the number of empty sets is the same as the structured number plus one. For example, the unstructured version of the natural number 2 is $\dot{2}$ which consist of two empty sets $\emptyset$ plus another empty set {} if both empty sets were removed from 2. Therefore, the unstructured number $\dot{2}$ consist of 3 empty sets. It follows therefore that from Table 4, a new number set called the "*unstructured number set*" can be defined as follows:

The unstructured number set U is a set that contains the elements $\dot{a}$ such that $\dot{a}$ represents the numbers of empty sets in the structured real number $'a'$ plus one.

The set of unstructured numbers is the key to division by zero. However, a relation between the natural number set and the unstructured number set must be formalized. This relation rest in the sequence of numbers in the last column of Table 4. This sequence of numbers is shown in Expression (1).

$$Sequence\ given\ in\ Table\ 3 = 1{,}2{,}3{,}5{,}9\ \ldots \quad (1)$$

The n^{th} term of the sequence in Expression (1) is given by Expression (2).

$$n^{th}term = \ 2^{n-1} + 1 \quad (2)$$

Expression (2) gives a partial relation between the structured and unstructured number set. This partial relation is given in Expression (3).

$$(2^{n-1} + 1) \times (0) = \dot{n} \quad (3)$$

Expression (3) indicates that multiplication by zero has meaning; this meaning is the conversion from the structured number n to the unstructured number $\dot{n}$. However, this relation would only work for natural numbers $n \geq 1$. For numbers less than 1 the relation is not adequate. Therefore, a more comprehensive relation $g(n)$ giving the relationship between any real number n and any unstructured number $\dot{n}$ is given in Expression (4).

$$g(n) = \begin{Bmatrix} sgn(n) \times |\dot{n}| = sgn(n) \times \left(2^{|n|-1} + 1\right) \times (0) & for\ \forall n \neq 0 \\ \dot{n} = 1 & for\ n = 0 \end{Bmatrix} \quad (4)$$

In Expression (4) $sgn(n)$ is simply the sign of the real number n. Examples of the use of the relation given in Expression (3) is given in Appendix 2. The relation given in Expression (4) shows that any real number a where $a = sgn(n) \times \left(2^{|n|-1} + 1\right)$ multiplied by zero will give an unstructured number $\dot{n}$. This solves the first discrepancy given in Table 3 as the relation given in Expression (4) implies the following:

$$a \times 0 = \dot{n}_1$$

$$b \times 0 = \dot{n}_2$$

This implies $a \times 0 \neq \mathrm{b} \times 0$

Where a and b are real numbers and $\dot{n}_1$ and $\dot{n}_2$ are unstructured numbers.

2.2. Graphical Interpretation and Representation

To graphically represent the relationship between the unstructured numbers and the structured real numbers consider the following graph shown in Figure 2.

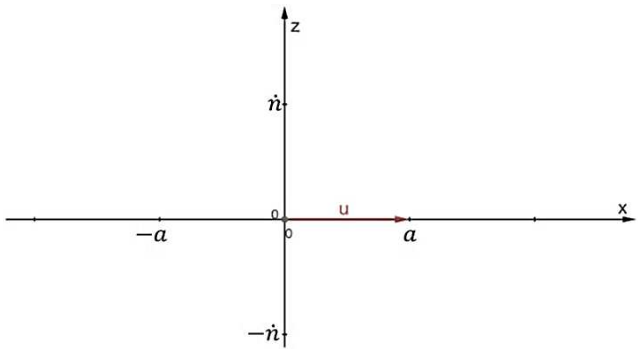

Figure 2. *Vector $\vec{u}$ with length "a" on the real x-axis.*

Figure 2 shows the xz-plane with the x-axis (representing the real numbers) and the z-axis (representing the unstructured numbers). The y-axis (representing the complex numbers) goes into the page.

From Figure 2, the vector $\vec{u}$ has length "a" where $a = sgn(n) \times (2^{n-1} + 1)$; here "$n$" is a real number. When "a" is multiplied by zero the result is $\dot{n}$; that is, $\dot{n} = a \times 0$. Graphically this is equivalent to rotating the vector $\vec{u}$, 90° counter-clockwise as shown in Figure 3. Here zero becomes a rotating factor of multiplication.

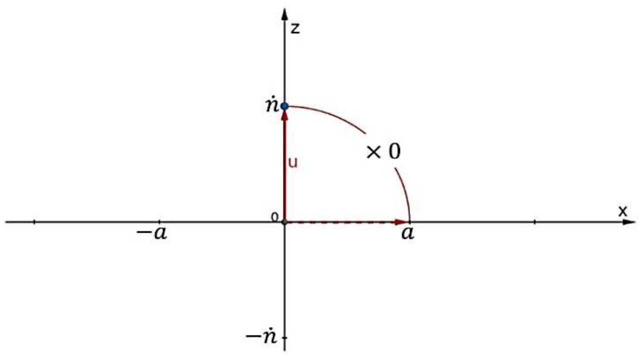

Figure 3. *Vector $\vec{u}$ with length "a" rotated 90° counter-clockwise by multiplying by rotational factor 0.*

Continuing the rotation involves multiplying $\dot{n}$ by $-\frac{1}{0}$ to get "$-a$"; that is:

$$\dot{n} \times -\frac{1}{0} = a \times 0 \times -\frac{1}{0}$$

$$= a \times -1 = -a$$

This can be represented by another 90° counter-clockwise as shown in Figure 4.

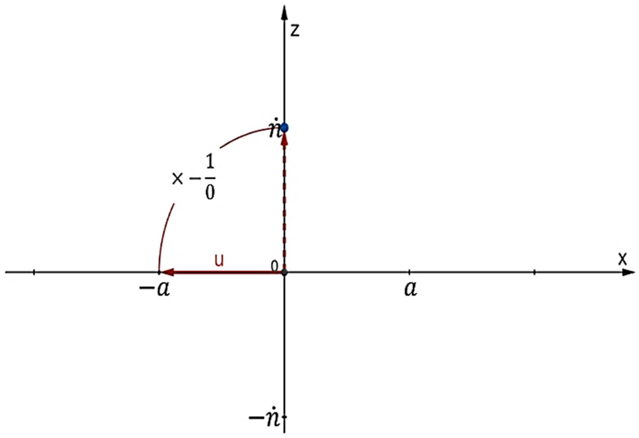

Figure 4. *Vector $\vec{u}$ with length "$\dot{n}$" rotated* 90° *counter-clockwise by multiplying by rotational factor* $-\frac{1}{0}$.

If we continue rotating counter-clockwise about the full xz-plane moving from "a" to $\dot{n}$ to "$-a$" to $-\dot{n}$ to and back to "a" again then the rotating factors of multiplication can be seen in Figure 5.

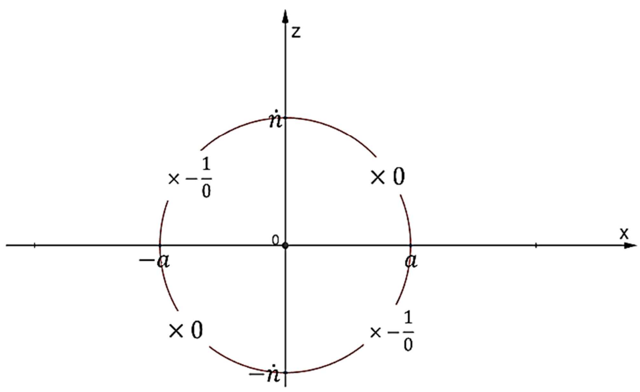

Figure 5. *Rotational factors for moving a vector $\vec{u}$ with length "a" counter-clockwise around the xz-plane.*

Hence to move from the real axis to the unstructured z-axis the real number must be multiplied by 0 and to move from the unstructured z-axis to the real x-axis the unstructured number must be multiplied by $-\frac{1}{0}$.

Just as rotating between the real and complex axis can be represented by multiplication of a single rotational unit i, rotation between the real and unstructured axis can be represented by multiplication of a single rotational unit.

2.3. Definition Unstructured Rotational Unit p

With complex numbers of the form $x + iy$, the rotational unit for the xy-plane is i, as shown in Figure 1. Hence for a real number a rotation about the real and complex axis involving the complex unit i would lead to the following equations:

$$a \times i = ai$$

$$a \times i^2 = -a$$

$$a \times i^3 = -ai$$

$$a \times i^4 = a \tag{5}$$

In a similar manner there needs to be a rotational unit that can be used to rotate between the x-axis and the z-axis and a similar unit that can be used to rotate between the y-axis and z-axis. For rotation between the z-axis and x-axis we defined a unit p such that:

$$a \times p = \dot{n}$$

$$a \times p^2 = -a$$

$$a \times p^3 = -\dot{n}$$

$$a \times p^4 = a \tag{6}$$

where

a Is a structured number such that $a = 2^{n-1} + 1$

$\dot{n}$ Unstructured number

There are a few things that are known about the powers of p. These are given in Expressions (7).

$$p = 0$$

$$p^2 = 0 \times \frac{-1}{0} = -1$$

$$p^3 = 0 \times \frac{-1}{0} \times 0 = -0$$

$$p^4 = 0 \times \frac{-1}{0} \times 0 \times \frac{-1}{0} = 1 \tag{7}$$

We can define the unit for the unstructured rotational unit as:

$$p^b = i^{b - f^b(0)} . f^b(1) \tag{8}$$

where

$f^b(c)$ Is a composite function such that $f(c) = 1 - c$

i Complex rotational unit

p Unstructured rotational unit

b An exponent belonging to the real number set

With the definition of p given in Equation (8), it can be shown that:

$$p = i^{1-f^1(0)} . f^1(1) = i^{1-1} . (0) = 0$$

$$p^2 = i^{2-f^2(0)} . f^2(1) = i^{2-0} . (1) = -1$$

$$p^3 = i^{3-f^3(0)} . f^3(1) = i^{3-1} . (0) = -0$$

$$p^4 = i^{4-f^4(0)} . f^4(1) = i^{4-0} . (0) = 1 \tag{9}$$

Hence since Expressions (9) evaluate to the same results as Expressions (7), then Equation (8) adequately represents the rotational unit whose multiplication results in rotations about the xz-plane, that is rotation between the real x-axis and the unstructured z -axis. Hence, powers of p provide the rotational values necessary to move from the real to the unstructured axis. Therefore, "p" can be formally defined as follows:

"p" is the unstructured rotational unit vector whose first power serves to:

(1) represent a $90°$ *counter-clockwise rotation along the real-unstructured plane (xz-plane) and*

(2) serves as a vector representation of the value of "zero".

Incidentally, with a small modification, this same unit can be used to move from the complex y-axis to the unstructured z-axis. To convert the complex number ai to an unstructured number (that is, to move around the complex unstructured yz-plane), the rotational unit becomes $-ip$. This leads to the following results:

$$ai \times -ip = \dot{n}$$

$$ai \times -ip^2 = -ai$$

$$ai \times -ip^3 = -\dot{n}$$

$$ai \times -ip^4 = ai \qquad (10)$$

It is important to note that since i and p are perpendicular vectors of rotation their dot product is zero. That is $ip = 0$.

2.4. Semi-structured Complex Numbers

Having already established the preliminaries in Sections 2.1 to 2.3, a new number can be defined to represent a point on the complete 3-dimensional space xyz-space created by the real, complex, and unstructured numbers as shown in Figure 6.

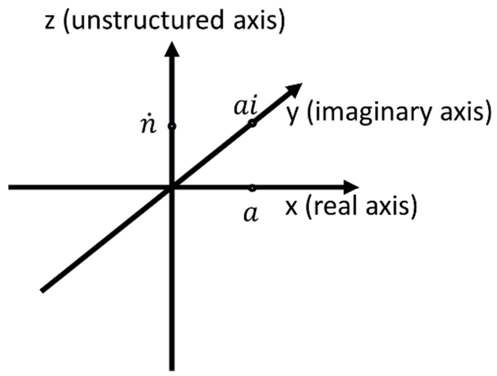

Figure 6. *3-dimensional space (xyz-space) formed by real, complex and unstructured axis.*

From Figure 6, semi-structured complex number can be defined as follows:

A semi-structured complex number is a number of the general form $s = a_r + b\phi_c + d\phi_u$; that is, a linear combination of real (ϕ_r), complex (ϕ_c) and unstructured (ϕ_u) rotational functions where a, b, c are real numbers.

The real (ϕ_r), complex (ϕ_c) and unstructured (ϕ_u) rotational functions are given in as follows:

Table 5. *Rotational functions for semi-structured complex number.*

$\phi_r = 1$
$\phi_c = i$
$\phi_u = p$

Given the values for the rotational functions as defined in Table 5, the number $r = a\phi_r + b\phi_c + d\phi_u$ can be written in the form:

$$s = a + bi + dp \qquad (11)$$

The number given in (11) is called semi-structured complex because it contains a structured complex part $(a + bi)$ and an unstructured part (dp). This semi-structured complex number can be used to solve equations that are undefined or indeterminate because of division by zero.

3. Satisfying the Field Axioms

To show that the semi-structured complex numbers are a complete ordered field and prove their existence, these numbers must satisfy the field axioms for numbered sets. These field axioms are provided in Appendix 1. The proof that these numbers satisfy the field axioms are given in Appendix 3. In satisfying the field axioms the authors of this paper legitimately have proven that the semi-structured complex number do in fact exist and agree with the normal rules of algebra.

With the introduction of semi-structured complex numbers and the algebra used to make calculations with these numbers, solutions to problems 1 to problem 3 given in Appendix 2 can be rewritten using semi-structure complex notation as shown in Appendix 4. Appendix 2 defines solutions to division by zero in purely unstructured numbers whereas Appendix 4 explains these same solutions in terms of rotations about the real-unstructured plane (xz-plane).

Having proven that semi-structured complex numbers do exist and can be used to solve division by zero problems, Appendix 5 provides a list of several research areas in which the use of semi-structured complex numbers could prove to be invaluable.

4. Conclusion

Division by zero has been a challenging issue in mathematics for centuries, without any concrete solution to what it means to divide by zero. The aim of this paper was to propose and prove the existence of a new number set in which division by zero is well defined.

An intuition of what division by zero realistically means was first developed and from intuitive perspective a number set called "The Semi-structured number set" ($\dot{S}$) was created. The existence was verified using axiomatic theory to show that the set is a "complete ordered field".

The number set was then used in Cartesian graphs to create a new axis called the Unstructured z-axis. The successful development of this proposed number set has implications not just in the field of mathematics but in other areas of science where mathematics is frequently used and division by zero is essential.

Appendix

Appendix 1. Field Axioms

Definition: A field is a nonempty set F containing at least 2 elements alongside the two binary operations of addition, $f_{+}: F \times F \to F$ such that $f_{+}(x, y) = x + y$ and multiplication $f(x, y) = x \cdot y$ that satisfy the following 11 axioms:

1. The operation of addition is closed, that is $\forall x \in F \; and \; \forall y \in F, x + y \in F$.
2. The operation of addition is commutative, that is $\forall x \in F, \forall y \in F, x + y = y + x$.
3. The operation of addition is associative, that $is \; \forall x \in F, \forall y \in F, \forall z \in F, x + (y + z) = (x + y) + z$.
4. The operation of addition has the additive identity element of 0 such that $\forall x \in F, x + 0 = x$
5. The operation of addition has the additive inverse element of $-x$ such that $\forall x \in F, x + (-x) = 0$.
6. The operation of multiplication is closed, that is $\forall x \in F, \forall y \in F, xy \in F$.
7. The operation of multiplication is commutative, that is $\forall x \in F, \forall y \in F, xy = yx$.
8. The operation of multiplication is associative, that is $\forall x \in F, \forall y \in F, \forall z \in F, x(yz) = (xy)z$.
9. The operation of multiplication has the multiplicative identity element of 1 such that $\forall x \in F, 1.x = x$.
10. The operation of multiplication has the multiplicative inverse element of $\frac{1}{x}$ such that $\forall x \in F, x \cdot \frac{1}{x} = 1$.
11. The operation of multiplication is distributive over addition, that is $\forall x \in F, \forall y \in F, \forall z \in F, x(y + z) = xy + xz$.

Appendix 2. Examples of the Use of Expression (3)

Consider the example of the use of unstructured and structured numbers:

Problem 1. What is the solution 4×0 *?*

Consider $(2^{n-1} + 1) \times (0) = \dot{n}$

$\to 4 \times 0 = (2^{n-1} + 1) \times 0$

$\to 4 = (2^{n-1} + 1)$

$\to 4 - 1 = 2^{n-1}$

$\to 3 = 2^{n-1}$

$\to \frac{\log 3}{\log 2} = n - 1$

$\to \frac{\log 3}{\log 2} + 1 = n$

$\to 1.5849 + 1 = n$

$\to 2.5849 = n$

Hence $4 \times 0 = 2.5\dot{8}49$

Interpretation: 4 zeros (empty sets) create an unstructured number $2.5\dot{8}49$

Problem 2: What is the solution $\frac{\dot{5}}{0}$ *?*

Consider $(2^{n-1} + 1) \times (0) = \dot{n}$

$\to \frac{\dot{5}}{0} = \frac{(2^{n-1}+1)\times 0}{0}$

$\to \frac{\dot{5}}{0} = (2^{n-1} + 1)$

$\to \frac{\dot{5}}{0} = 2^{5-1} + 1$

$\to \frac{\dot{5}}{0} = 16 + 1$

$\to \frac{\dot{5}}{0} = 17$

Hence $\frac{\dot{5}}{0} = 17$

Interpretation: The unstructured number $\dot{5}$ contains enough zeros to create the structured real number 17

Problem 3: What is the solution $\frac{9}{0}$ *? (please read section 2.2 to 2.4 before attempting this problem)*

Consider $(2^{n-1} + 1) \times (0) = \dot{n}$

$\to (2^{n-1} + 1) = \frac{\dot{n}}{0}$

$\to 9 = \frac{\dot{n}_1}{0}$ and $0 = \frac{\dot{n}_2}{0}$

$\rightarrow 9 = (2^{4-1} + 1) = \frac{\dot{4}}{0}$ and $0 = \frac{\dot{1}}{0}$

$\rightarrow \frac{9}{0} = \frac{\dot{4}}{0} \div \frac{\dot{1}}{0}$

$\rightarrow \frac{9}{0} = \frac{\dot{4}}{0} \times \frac{0}{\dot{1}}$

$\rightarrow \frac{9}{0} = \frac{\dot{4}}{\dot{1}} = \dot{4}$

Interpretation: The structured number 9 contains enough zeros to create the semi-structured complex number $\dot{4}$

Appendix 3. Proof That the Set of Semi-structured Complex Numbers $\mathbb{S}$ satisfies the field Axioms properties

Axiom 1: The operation of addition is closed, that is $\forall\, \boldsymbol{h_1} \in \mathbb{S}\ \boldsymbol{and}\ \forall\, \boldsymbol{h_2} \in \mathbb{S},\ \boldsymbol{h_1} + \boldsymbol{h_2} \in \mathbb{S}.$

Consider $h_1 = (x_1 + y_1 i + u_1\mathrm{p}) \in \mathbb{S}\ and\ h_2 = (x_2 + y_2 i + u_2\mathrm{p}) \in \mathbb{S}.$

$$\mathbf{h_1} + \mathbf{h_2} = (x_1 + y_1 i + u_1\mathrm{p}) + (x_2 + y_2 i + u_2\mathrm{p}) = (x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p}$$

Since $x_1, x_2, y_1, y_2, z_1\ and\ z_2$ are all real numbers, then $(x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p} \in \mathbb{S}$. Hence, unstructured complex number set $\mathbb{S}$ is closed under addition.

Axiom 2: The operation of addition is commutative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1 + h_2 = h_2 + h_1.$

Consider $h_1 = (x_1 + y_1 i + u_1\mathrm{p}) \in \mathbb{S}\ and\ h_2 = (x_2 + y_2 i + u_2\mathrm{p}) \in \mathbb{S}.$

$$h_1 + h_2 = (x_1 + y_1 i + u_1\mathrm{p}) + (x_2 + y_2 i + u_2\mathrm{p})$$

$$h_1 + h_2 = (x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p}$$

$$h_1 + h_2 = (x_2 + x_1) + (y_2 + y_1)i + (u_2 + u_1)\mathrm{p}$$

$$h_1 + h_2 = (x_2 + y_2 i + u_2\mathrm{p}) + (x_1 + y_1 i + u_1\mathrm{p})$$

But

$$h_2 + h_1 = (x_2 + y_2 i + u_2\mathrm{p}) + (x_1 + y_1 i + u_1\mathrm{p})$$

Hence

$$h_1 + h_2 = h_2 + h_1$$

Therefore, since $h_1 + h_2 = h_2 + h_1$. Consequently, unstructured complex number set $\mathbb{S}$ is commutative under addition.

Axiom 3: The operation of addition is associative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S}, \forall\, h_3 \in \mathbb{S},\ h_1 + (h_2 + h_3) = (h_1 + h_2) + h_3.$

Consider

$$h_1 = (x_1 + y_1 i + u_1\mathrm{p}) \in \mathbb{S};\ h_2 = (x_2 + y_2 i + u_2\mathrm{p}) \in \mathbb{S} \text{ and } h_3 = (x_3 + y_3 i + u_3\mathrm{p}) \in \mathbb{S}.$$

$$h_1 + (h_2 + h_3) = (x_1 + y_1 i + u_1\mathrm{p}) + [(x_2 + y_2 i + u_2\mathrm{p}) + (x_3 + y_3 i + u_3\mathrm{p})]$$

$$h_1 + (h_2 + h_3) = (x_1 + y_1 i + u_1\mathrm{p}) + [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}]$$

$$h_1 + (h_2 + h_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3)i + (u_1 + u_2 + u_3)\mathrm{p}$$

$$h_1 + (h_2 + h_3) = [(x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p}] + (x_3 + y_3 i + u_3\mathrm{p})$$

But

$$(h_1 + h_2) + h_3 = [(x_1 + x_2) + (y_1 + y_2)i + (u_1 + u_2)\mathrm{p}] + (x_3 + y_3 i + u_3\mathrm{p})$$

Hence

$$h_1 + (h_2 + h_3) = (h_1 + h_2) + h_3$$

Hence segmental number set $\mathbb{S}$ is associative under addition.

Axiom 4: The operation of addition has the additive identity element of 0 *such that* $\forall h \in \mathbb{S}, h + 0 = h.$

Consider $h_1 = (x_1 + y_1 i + u_1\mathrm{p}) \in \mathbb{S}\ and\ h_2 = (0 + 0i + 0\mathrm{p}) \in \mathbb{S}.$

$$h_1 + 0 = (x_1 + y_1 i + u_1\mathrm{p}) + (0 + 0i + 0\mathrm{p})$$

$$h_1 + 0 = (x_1 + 0) + (y_1 + 0)i + (u_1 + 0)p$$

Since x_1, y_1, u_1 *and* 0 are real numbers, then $x_1 + 0 = x_1$; $y_1 + 0 = y_1$; and $u_1 + 0 = u_1$. Hence

$$h_1 + 0 = x_1 + y_1 i + u_1 \mathrm{p}$$

$$h_1 + 0 = h_1$$

Consequently, unstructured complex number set $\mathbb{S}$ has the additive identity element of 0.

Axiom 5: The operation of addition has the additive inverse element of $-h$ *such that* $\forall h \in \mathbb{S}, h + (-h) = 0$.

Consider $h = (x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$ *and* $-h = -(x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$.

Now $-h = -(x_1 + y_1 i + u_1 \mathrm{p}) = (-x_1 - y_1 i - u_1 \mathrm{p})$

This implies:

$$h + (-h) = x_1 + y_1 i + u_1 \mathrm{p} - x_1 - y_1 i - u_1 \mathrm{p}$$

$$h + (-h) = (x_1 - x_1) + (y_1 - y_1) i + (u_1 - u_1) p$$

$$h + (-h) = 0 + 0i + 0p = 0$$

Hence the unstructured complex number set $\mathbb{S}$ has the additive inverse element of $-h$ such that $\forall h \in \mathbb{S}, h + (-h) = 0$

Axiom 6: T*he operation of multiplication is closed, that is* $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1\, h_2 \in \mathbb{S}$.

Consider $h_1 = (x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$ *and* $h_2 = (x_2 + y_2 i + u_2 \mathrm{p}) \in \mathbb{S}$.

$$h_1\, h_2 = (x_1 + y_1 i + u_1 \mathrm{p}) \times (x_2 + y_2 i + u_2 \mathrm{p})$$

$$h_1\, h_2 = x_1 \times\ (x_2 + y_2 i + u_2 \mathrm{p}) + y_1 i \times\ (x_2 + y_2 i + u_2 \mathrm{p}) + u_1 \mathrm{p} \times\ (x_2 + y_2 i + u_2 \mathrm{p})$$

$$h_1\, h_2 = x_1 x_2 + x_1 y_2 i + x_1 u_2 \mathrm{p} + y_1 x_2 i + y_1 y_2 ii + y_1 u_2 \mathrm{ip} + u_1 x_2 \mathrm{p} + u_1 y_2 ip + u_1 u_2 \mathrm{pp}$$

Now $ii = i^2 = -1$; $ip = 0$; and $\mathrm{pp} = p^2 = -1$.

One important note here is that since i and p are perpendicular rotational vectors then their dot product is zero that is $ip = 0$; Hence

$$h_1\, h_2 = x_1 x_2 + x_1 y_2 i + x_1 u_2 \mathrm{p} + y_1 x_2 i - y_1 y_2 + u_1 x_2 \mathrm{p} - u_1 u_2$$

$$h_1\, h_2 = x_1 x_2 - y_1 y_2 - u_1 u_2 + x_1 y_2 i + y_1 x_2 i + x_1 u_2 \mathrm{p} + u_1 x_2 \mathrm{p}$$

$$h_1\, h_2 = (x_1 x_2 - y_1 y_2 - u_1 u_2) + (x_1 y_2 + y_1 x_2) i + (x_1 u_2 + u_1 x_2) p$$

Consequently, for the unstructured complex number set $\mathbb{S}$, the operation of multiplication is closed, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1\, h_2 \in \mathbb{S}$.

Axiom 7: The operation of multiplication is commutative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1\, h_2 = h_2\, h_1$.

Consider $h_1 = (x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$ *and* $h_2 = (x_2 + y_2 i + u_2 \mathrm{p}) \in \mathbb{S}$. From axiom 6,

$$h_1\, h_2 = (x_1 x_2 - y_1 y_2 - u_1 u_2) + (x_1 y_2 + y_1 x_2) i + (x_1 u_2 + u_1 x_2) p$$

Now

$$h_2\, h_1 = (x_2 + y_2 i + u_2 \mathrm{p}) \times (x_1 + y_1 i + u_1 \mathrm{p})$$

$$h_2\, h_1 = x_2 \times\ (x_1 + y_1 i + u_1 \mathrm{p}) + y_2 i \times\ (x_1 + y_1 i + u_1 \mathrm{p}) + u_2 \mathrm{p} \times\ (x_1 + y_1 i + u_1 \mathrm{p})$$

$$h_2\, h_1 = x_2 x_1 + x_2 y_1 i + x_2 u_1 \mathrm{p} + y_2 x_1 i + y_2 y_1 ii + y_2 u_1 \mathrm{ip} + u_2 x_1 \mathrm{p} + u_2 y_1 ip + u_2 u_1 \mathrm{pp}$$

Now $ii = i^2 = -1$; $ip = 0$; and $\mathrm{pp} = p^2 = -1$.

One important not here is that since i and p are perpendicular rotational vectors then their dot product is zero that is $ip = 0$; Hence

$$h_2\, h_1 = x_2 x_1 + x_2 y_1 i + x_2 u_1 \mathrm{p} + y_2 x_1 i - y_2 y_1 + u_2 x_1 \mathrm{p} - u_2 u_1$$

$$h_2\, h_1 = x_2 x_1 - y_2 y_1 - u_2 u_1 + x_2 y_1 i + y_2 x_1 i + x_2 u_1 \mathrm{p} + u_2 x_1 \mathrm{p}$$

$$h_2\, h_1 = (x_2 x_1 - y_2 y_1 - u_2 u_1) + (x_2 y_1 + y_2 x_1) i + (x_2 u_1 + u_2 x_1) p$$

It is clear that $h_1\, h_2 = h_2\, h_1$. Consequently, for the unstructured complex number set $\mathbb{S}$, the operation of multiplication is commutative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S},\ h_1\, h_2 = h_2\, h_1$.

Axiom 8: The *operation of multiplication is associative, that is* $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S}, \forall\, h_3 \in \mathbb{S},\ h_1 (\, h_2\, h_3) = (\, h_1\, h_2)\, h_3$.

Consider $h_1 = (x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$; $h_2 = (x_2 + y_2 i + u_2 \mathrm{p}) \in \mathbb{S}$; $h_3 = (x_3 + y_3 i + u_3 \mathrm{p}) \in \mathbb{S}$.

$$h_1(\,h_2\,h_3) = (x_1 + y_1 i + u_1 \mathrm{p}) \times [(x_2 + y_2 i + u_2 \mathrm{p}) \times (x_3 + y_3 i + u_3 \mathrm{p})]$$

$$h_1(\,h_2\,h_3) = (x_1 + y_1 i + u_1 \mathrm{p}) \times [(x_2 x_3 - y_2 y_3 - u_2 u_3) + (x_2 y_3 + y_2 x_3)i + (x_2 u_3 + u_2 x_3)p]$$

$$h_1(\,h_2\,h_3) = [x_1.(x_2 x_3 - y_2 y_3 - u_2 u_3) - y_1.(x_2 y_3 + y_2 x_3) - u_1.(x_2 u_3 + u_2 x_3)] + [x_1.(x_2 y_3 + y_2 x_3) + y_1.(x_2 x_3 - y_2 y_3 - u_2 u_3)]i + [x_1.(x_2 u_3 + u_2 x_3) + u_1.(x_2 x_3 - y_2 y_3 - u_2 u_3)]p$$

Now consider:

$$(\,h_1\,h_2)\,h_3 = [(x_1 x_2 - y_1 y_2 - u_1 u_2) + (x_1 y_2 + y_1 x_2)i + (x_1 u_2 + u_1 x_2)p] \times (x_3 + y_3 i + u_3 \mathrm{p})$$

$$(\,h_1\,h_2)\,h_3 = [x_3.(x_1 x_2 - y_1 y_2 - u_1 u_2) - y_3.(x_1 y_2 + y_1 x_2) - u_3.(x_1 u_2 + u_1 x_2)] + [x_3.(x_1 y_2 + y_1 x_2) + y_3.(x_1 x_2 - y_1 y_2 - u_1 u_2)]i + [x_3.(x_1 u_2 + u_1 x_2) + u_3.(x_1 x_2 - y_1 y_2 - u_1 u_2)]p$$

With a bit more calculation it can be seen:

$$[x_1.(x_2 x_3 - y_2 y_3 - u_2 u_3) - y_1.(x_2 y_3 + y_2 x_3) - u_1.(x_2 u_3 + u_2 x_3)] = [x_3.(x_1 x_2 - y_1 y_2 - u_1 u_2) - y_3.(x_1 y_2 + y_1 x_2) - u_3.(x_1 u_2 + u_1 x_2)]$$

$$[x_1.(x_2 y_3 + y_2 x_3) + y_1.(x_2 x_3 - y_2 y_3 - u_2 u_3)]i = [x_3.(x_1 y_2 + y_1 x_2) + y_3.(x_1 x_2 - y_1 y_2 - u_1 u_2)]i$$

$$[x_1.(x_2 u_3 + u_2 x_3) + u_1.(x_2 x_3 - y_2 y_3 - u_2 u_3)]p = [x_3.(x_1 u_2 + u_1 x_2) + u_3.(x_1 x_2 - y_1 y_2 - u_1 u_2)]p$$

It is clear that $h_1(\,h_2\,h_3) = (\,h_1\,h_2)\,h_3$. Consequently, for the unstructured complex number set $\mathbb{S}$ the operation of multiplication is associative, that is $\forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S}, \forall\, h_3 \in \mathbb{S},\ h_1(\,h_2\,h_3) = (\,h_1\,h_2)\,h_3$.

Axiom 9: The operation of multiplication has the multiplicative identity element of 1 *such that* $\forall h \in \mathbb{S}, 1.h = h$.

Consider $h = (x + yi + up) \in \mathbb{S}\ and\ 1 = (1 + 0i + 0\mathrm{p}) \in \mathbb{S}$.

$$1.x = (1 + 0i + 0\mathrm{p}) \times (x + yi + up)$$

Now from Axiom 6.

$$1.x = (1.x - 0.y - 0.u) + (1.y + y.0)i + (1.u + 0.\mathrm{x})p$$

$$1.x = (1.x) + (1.y)i + (1.u)p$$

$$1.x = x$$

Consequently, for the segmental complex number set $\mathbb{S}$ has the multiplicative identity element of 1 such that $\forall h \in \mathbb{S}, 1.h = h$.

Axiom 10: The operation of multiplication has the multiplicative inverse element of $\frac{1}{h}$ *such that* $\forall h \in \mathbb{S}, h \cdot \frac{1}{h} = 1$.

It needs to be shown that for $h = (x + yi + up) \in \mathbb{S}$ there exist a $\frac{1}{h} = (q + ri + sp) \in \mathbb{S}$ such that $h \cdot \frac{1}{h} = 1$. From Axiom 6,

$$h \cdot \frac{1}{h} = (xq - yr - us) + (xr + yq)i + (xs + uq)p$$

Now suppose $h \cdot \frac{1}{h} = 1$, this implies:

$$1 + 0.i + 0.j = (xq - yr - us) + (xr + yq)i + (xs + uq)p$$

If two semi-structured complex numbers are equal, then their real parts, imaginary parts and their unstructured parts must be equal. This results in the following simultaneous equations:

$xq - yr - us = 1$ $\qquad$ $xr + yq = 0$ $\qquad$ $xs + uq = 0$

The solution to these simultaneous equations gives:

$q = \frac{x}{x^2+y^2+u^2}$ $\qquad$ $r = \frac{-y}{x^2+y^2+u^2}$ $\qquad$ $s = \frac{-u}{x^2+y^2+u^2}$

So, the reciprocal of $x = (a + bi + cj)$ is the number $\frac{1}{x} = (q + ri + sp)$ where q, r and s have the values just found. In summary, we have the following reciprocation formula:

$$\frac{1}{(x+yi+up)} = \left(\frac{x}{x^2+y^2+u^2}\right) + \left(\frac{-y}{x^2+y^2+u^2}\right)i + \left(\frac{-u}{x^2+y^2+u^2}\right)p$$

Consequently, for the segmental complex number set $\mathbb{S}$ the operation of multiplication has the multiplicative inverse *element of* $\frac{1}{h}$ *such that* $\forall h \in \mathbb{S}, h \cdot \frac{1}{h} = 1$.

Axiom 11: The operation of multiplication is distributive over addition, that $is \, \forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S}, \forall\, h_3 \in \mathbb{S},\ h_1 \times (h_2 + h_3) = h_1 h_2 + h_1 h_3$.

Consider $h_1 = (x_1 + y_1 i + u_1 \mathrm{p}) \in \mathbb{S}$; $h_2 = (x_2 + y_2 i + u_2 \mathrm{p}) \in \mathbb{S}$ and $h_3 = (x_3 + y_3 i + u_3 \mathrm{p}) \in \mathbb{S}$.

$$h_1 \times (h_2 + h_3) = (x_1 + y_1 i + u_1 \mathrm{p}) \times [(x_2 + y_2 i + u_2 \mathrm{p}) + (x_3 + y_3 i + u_3 \mathrm{p})]$$

$$h_1 \times (h_2 + h_3) = (x_1 + y_1 i + u_1 \mathrm{p}) \times [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}]$$

$$h_1 \times (h_2 + h_3) = (x_1 + y_1 i + u_1 \mathrm{p}) \times [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}]$$

$$h_1 \times (h_2 + h_3) = x_1 \times [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}] + y_1 i \times [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}] + u_1 \mathrm{p} \times [(x_2 + x_3) + (y_2 + y_3)i + (u_2 + u_3)\mathrm{p}]$$

$$h_1 \times (h_2 + h_3) = [x_1(x_2 + x_3) - y_1(y_2 + y_3) - u_1(u_2 + u_3)] + [x_1(y_2 + y_3) + y_1(x_2 + x_3)]i + [x_1(u_2 + u_3) + u_1(x_2 + x_3)]\mathrm{p}$$

Now from Axiom 6

$$h_1 h_2 + h_1 h_3 = [(x_1x_2 - y_1y_2 - u_1u_2) + (x_1y_2 + y_1x_2)i + (x_1u_2 + u_1x_2)p] + [(x_1x_3 - y_1y_3 - u_1u_3) + (x_1y_3 + y_1x_3)i + (x_1u_3 + u_1x_3)p]$$

$$h_1 h_2 + h_1 h_3 = [(x_1x_2 - y_1y_2 - u_1u_2 + x_1x_3 - y_1y_3 - u_1u_3) + (x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3)i + (x_1u_2 + u_1x_2 + x_1u_3 + u_1x_3)p]$$

With a bit more calculation it is clear that:

$$x_1(x_2 + x_3) - y_1(y_2 + y_3) - u_1(u_2 + u_3) = x_1x_2 - y_1y_2 - u_1u_2 + x_1x_3 - y_1y_3 - u_1u_3$$

$$x_1(y_2 + y_3) + y_1(x_2 + x_3) = x_1y_2 + y_1x_2 + x_1y_3 + y_1x_3$$

$$x_1(u_2 + u_3) + u_1(x_2 + x_3) = x_1u_2 + u_1x_2 + x_1u_3 + u_1x_3$$

Since the coefficients of $h_1 \times (h_2 + h_3)$ and $h_1 h_2 + h_1 h_3$, are the same, this implies that the operation of multiplication is distributive over addition, that $is \, \forall\, h_1 \in \mathbb{S}, \forall\, h_2 \in \mathbb{S}, \forall\, h_3 \in \mathbb{S},\ h_1 \times (h_2 + h_3) = h_1 h_2 + h_1 h_3$

Appendix 4. Solution to Division by Zero Using Semi-structured Complex Numbers

Consider the example of the use of unstructured and structured numbers:

Problem 1: What is the solution $\times 0$ *?*

Recall that p is the unstructured rotational vector whose first power evaluates to zero. Therefore 4×0 can be rewritten as the semi-structured number $4p$. That is:

$4 \times 0 = 4p$

Interpretation: 4 multiplied by zero is equivalent to the 90^o counter-clockwise rotation along the (real-unstructured plane) xz-plane of a vector 4 times the magnitude of the unit unstructured rotational vector p.

Problem 2: What is the solution $\frac{9}{0}$ *? (please read section 2.2 to 2.4 before attempting this problem*)

Consider $\frac{9}{0} = 9 \times \frac{1}{0}$

$\rightarrow \frac{9}{0} = 9 \times \frac{1}{0}$

$\rightarrow \frac{9}{0} = 9 \times \frac{p^4}{p}$

$\rightarrow \frac{9}{0} = 9 \times p^3$

But $p^3 = p^2 \times p = -p$

$\rightarrow \frac{9}{0} = -9p$

Interpretation: 9 divided by zero is equivalent to the 90^o clockwise rotation along the (real-unstructured plane) xz-plane of a vector 9 times the magnitude of the unit unstructured rotational vector p.

Problem 3: What is the solution $\frac{\dot{5}}{0}$ *?*

Consider $(2^{n-1}+1)\times(0)=\dot{n}$

$$\rightarrow \frac{\dot{5}}{0}=\frac{(2^{n-1}+1)\times 0}{0}$$
$$\rightarrow \frac{\dot{5}}{0}=(2^{n-1}+1)=2^{5-1}+1$$
$$\rightarrow \frac{\dot{5}}{0}=17$$
$$\rightarrow \frac{\dot{5}}{0}=17\times 1$$
$$\rightarrow \frac{\dot{5}}{0}=17p^4 \text{ (since } p^4=1)$$

Hence $\frac{\dot{5}}{0}=17p^4$

Interpretation: The unstructured number $\dot{5}$ divided by zero is equivalent to the 360^o counter-clockwise rotation along the (real-unstructured plane) xz-plane of a vector 17 times the magnitude of the unit unstructured rotational vector p.

Appendix 5. Research Areas in Which Semi-structured Complex Numbers Would Be Useful

Table 6 provides some areas of research that semi-structured complex numbers can be useful. Whilst this list is not exhaustive it does point to the fact that semi-structured complex numbers can play an important role in the development of several areas of mathematics.

Table 6. *Research areas in which division by zero and semi-structured complex numbers may be useful.*

Research Area	Reference
Division by zero calculus	[6-8]
Multispectral hyperspaces	[9, 10]
Pappus Chain theorem	[11]
Wasan geometry	[12, 13]
Probability and Stochastic Analysis	[14, 15]
Control theory	[16]

References

[1] J. Stillwell, *Reverse mathematics: proofs from the inside out*. Princeton University Press, 2019.

[2] "Division by zero," *[Online] Avaliable at: https://en.wikipedia.org/wiki/Division_by_zero#cite_ref-2. [Retirved 31 December 2020]*.

[3] H. Okumura and S. Saitoh, "Wasan Geometry and Division by Zero Calculus," *Sangaku Journal of Mathematics (SJM),* vol. 2, pp. 57-73, 2018.

[4] T. S. dos Reis, W. Gomide, and J. A. Anderson, "Construction of the Transreal Numbers and Algebraic Transfields," *International Journal of Applied Mathematics,* vol. 46, no. 1, 2016.

[5] C. Dunne, J. B. Wells, and F. Kamareddine, "Adding an Abstraction Barrier to ZF Set Theory," in *International Conference on Intelligent Computer Mathematics*, 2020, pp. 89-104: Springer.

[6] H. Okumura and S. Saitoh, "The Descartes circles theorem and division by zero calculus," arXiv preprint arXiv: 1711.04961, 2017.

[7] S. SAITOH, Introduction to the Division by Zero Calculus. Scientific Research Publishing, Inc. USA, 2021.

[8] UFUOMA, O. (2018). The differential and integral calculus in Bhaskara's framework: Exploration of zero and infinity. Asian Journal of Mathematics and Computer Research, 511-550.

[9] J. Czajko, "Algebraic division by zero implemented as quasigeometric multiplication by infinity in real and complex multispatial hyperspaces," World Scientific News, vol. 92, no. 2, pp. 171-197, 2018.

[10] J. Czajko, "Quaternionic division by zero is implemented as multiplication by infinity in 4D hyperspace," World Scientific News, vol. 94, no. 2, pp. 190-216, 2018.

[11] H. Okumura, "An Analogue to Pappus Chain theorem with Division by Zero," in Forum Geom, 2018, vol. 18, pp. 409-412.

[12] H. Okumura, "Wasan geometry with the division by 0," arXiv preprint arXiv: 1711.06947, 2017.

[13] H. Okumura and S. Saitoh, "Applications of the division by zero calculus to Wasan geometry," GLOBAL JOURNAL OF ADVANCED RESEARCH ON CLASSICAL AND MODERN GEOMETRIES"(GJARCMG), vol. 7, no. 2, pp. 44-49, 2018.

[14] T. Matsuura, H. Okumura, and S. Saitoh, "Probability and Stochastic Analysis in Reproducing Kernels and Division by Zero Calculus," 2020.

[15] W. P. Mwangi, "Definite Probabilities from Division of Zero by Itself Perspective," Asian Journal of Probability and Statistics, pp. 1-26, 2020.

[16] Y. Zhang, J. Guo, D. Zhang, B. Qiu, and Z. Yang, "Output tracking of time-varying linear system using ZD controller with pseudo division-by-zero phenomenon illustrated," in IECON 2017-43rd Annual Conference of the IEEE Industrial Electronics Society, 2017, pp. 3075-3080: IEEE.