

Some Finiteness Conditions for Strong Semilattice of Semigroups

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Abstract: Let $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ be a strong semilattice of semigroups such that I is finite and each S_i ($i \in I$) be a family of disjoint semigroups. In this article some finiteness conditions which are periodicity, local finiteness and locally finite presentability are considered for S . It is proven that a strong semilattice of semigroups $\mathcal{S}[I; S_i, \phi_{j,i}]$ is periodic, locally finite, locally finitely presented and residually finite, respectively if and only if I is finite and each semigroup S_i ($i \in I$) is periodic, locally finite, locally finitely presented and residually finite, respectively.

Keywords: Strong Semilattice of Semigroups, Finiteness Conditions, Periodicity, Local Finiteness, Locally Finite Presentability, Residual Finiteness

1. Introduction

The study of finiteness conditions for semigroups (the properties of semigroups which all finite semigroups have) is one of the most important topics for the mathematical community. Some of the properties of finiteness are being finitely generated, finite presentability, having a soluble word problem, periodicity, local finiteness, local finite presentability, residual finiteness, being hopfian, having finite complete rewriting systems and finite derivation type (FDT). They have been considered for certain classes of semigroup constructions (see [1-9]). In [1], it was shown that the properties of being finite, being finitely generated, being finitely presented and being residually finite are preserved by large subsemigroups and small extensions. In [5], periodicity, local finiteness, residual finiteness of Rees matrix semigroups and solvable word problem for Rees matrix semigroups were investigated.

One of semigroup structures is a strong semilattice of semigroups which is an important structure for completely regular semigroups (see [10]). Finite presentability of strong semilattice of semigroups was investigated in [3]. The authors of [3] first proved that a band of semigroups S_i ($i \in I$) is finitely presented if I is finite and each S_i is

finitely presented (see [3, Corollary 4.5]). Moreover, they proved that a band of monoids S_i ($i \in I$) is finitely presented if and only if I is finite and each S_i is finitely presented (see [3, Corollary 5.6]). Then, from Theorem 1.3 in [1] they concluded that a strong semilattice of semigroups $\mathcal{S}[I; S_i, \phi_{j,i}]$ is finitely presented if each S_i ($i \in I$) is finitely presented and if I is finite. In [4] the authors constructed more specific presentations for strong semilattice of semigroups from given presentations for S_i ($i \in I$) and a presentation for each S_i from a given presentation for $\mathcal{S}[I; S_i, \phi_{j,i}]$. Also they proved that if $\mathcal{S}[I; S_i, \phi_{j,i}]$ is finitely generated then $\mathcal{S}[I; S_i, \phi_{j,i}]$ has a soluble word problem if and only if each S_i has a soluble word problem. In [9] the author showed that a strong semilattice of semigroups $\mathcal{S}[I; S_i, \phi_{j,i}]$ has FDT if and only if I is finite and every semigroup S_i ($i \in I$) has FDT.

In this paper periodicity, locally finiteness, locally finite presentability and residual finiteness of strong semilattice of semigroups are investigated.

2. Notation and Basic Definitions

Let \mathcal{P} be any property of semigroups. It said that a semigroup is locally \mathcal{P} if every finitely generated subsemigroup of has property \mathcal{P} . If \mathcal{P} is a finiteness condition, then being locally \mathcal{P} is also a finiteness condition. In particular, being locally finite and being locally finitely presented are finiteness conditions.

Let I be a semilattice, and let S_i ($i \in I$) be a family of disjoint semigroups. Suppose that, for any two elements $i, j \in I$, with $i \leq j$ ($i \leq j$ if and only if $ij = i$), there is a homomorphism $\phi_{j,i} : S_j \rightarrow S_i$, and that these homomorphisms satisfy the following conditions:

1. for each $i \in I$, $\phi_{i,i}$ is the identity map 1_{S_i} ;
2. for all $i, j, k \in I$, such that $i \leq j \leq k$ $\phi_{k,j}\phi_{j,i} = \phi_{k,i}$.

Define a multiplication on $S = \bigcup_{i \in I} S_i$ in terms of the multiplications in the components S_i and the homomorphisms $\phi_{j,i}$, by the rule that, for each $x \in S_i$ and $y \in S_j$,

$$xy = (x\phi_{i,ij})(y\phi_{j,ij})$$

With respect to this multiplication, S is a semigroup, called a strong semilattice of semigroups, and is denoted by $\mathcal{S}[I; S_i, \phi_{j,i}]$.

Let A be an alphabet, A^+ be the free semigroup on A (i.e. the set of all non-empty words over A). A semigroup presentation is an ordered pair $\langle A | R \rangle$ where $R \subseteq A^+ \times A^+$. An element a of A is called a generating symbol, while an element (r, s) of R is called a defining relation, and is usually written as $r = s$. Also if $A = \{a_1, a_2, \dots, a_n\}$ and $R = \{r_1 = s_1, r_2 = s_2, \dots, r_n = s_n\}$, we write $\langle a_1, a_2, \dots, a_n | r_1 = s_1, r_2 = s_2, \dots, r_n = s_n \rangle$ for $\langle A | R \rangle$. The semigroup defined by a presentation $\langle A | R \rangle$ is A^+ / ρ , where ρ is the least congruence on A^+ containing R . More generally, a semigroup S is said to be defined by the presentation $\langle A | R \rangle$ if $S \cong A^+ / \rho$. Thus, elements of S are in one-one correspondence with the congruence classes of words from A^+ , or, to put it differently, each word from A^+ represents an element of S . As usual, for any two words $u_1, u_2 \in A^+$ we write $u_1 = u_2$ if $u_1 \rho = u_2 \rho$ (i.e. if they represent the same element in S), and $u_1 \equiv u_2$ if they are

identical words. The semigroup S is finitely generated if the set A is finite; S is finitely presented if both sets A and R are finite.

3. Periodicity

Recall that a semigroup S is *periodic* if, for each $s \in S$, the monogenic semigroup generated by s is finite, or equivalently there exist distinct positive integers m and r (depending on s) such that $s^{m+r} = s^m$.

Theorem 3.1 Let I be a finite semilattice, and let S_i ($i \in I$) be a family of disjoint semigroups. The strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is periodic if and only if, for each $i \in I$, S_i is periodic.

Proof.

(\Rightarrow) Since each S_i ($i \in I$) is a subsemigroup of the strong semilattice S and the subsemigroups of the periodic semigroups are periodic, each semigroup S_i ($i \in I$) is periodic.

(\Leftarrow) Let each S_i ($i \in I$) be periodic. Let $s \in \bigcup_{i \in I} S_i$. Then, there exists an i such that $s \in S_i$. Since S_i is periodic, there exist positive integers m and r such that $s^{m+r} = s^m$. So the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is periodic.

4. Local Finiteness

Let X be any non-empty subset of a semigroup S . Then the smallest subsemigroup of S containing X is called the subsemigroup of S generated by X , and is denoted by $\langle X \rangle$. If X is finite then $\langle X \rangle$ is called a finitely generated subsemigroup. If every finitely generated subsemigroup of S is finite, then S is called *locally finite*.

Theorem 4.1

Let I be a finite semilattice, and let S_i ($i \in I$) be a family of disjoint semigroups. The strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is locally finite if and only if, for each $i \in I$, S_i is locally finite.

Proof.

(\Rightarrow) Since each S_i ($i \in I$) is a subsemigroup of the strong semilattice S and the subsemigroups of the locally finite semigroups are locally finite, each semigroup S_i ($i \in I$) is locally finite.

(\Leftarrow) Let each S_i ($i \in I$) be locally finite. Let X be a finite subset of the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. Now it is shown that $\langle X \rangle$, the subsemigroup generated by X , is finite.

Let $J \subseteq I$ and define J as follows

$$i \in J \Leftrightarrow X \cap S_i \neq \emptyset \quad (i \in I)$$

Then, since X is finite, the set $J = \{i_1, \dots, i_n\}$ is finite. Let $X_i = X \cap S_i$ for each $i \in J$. Then $\{X_i\}_{i \in J}$ is a partition of X . Also, for each subset $\{j_1, \dots, j_k\}$ of J , there exists a λ such that $j_1 \cdots j_k \leq j_\lambda$. Then, since $1 \leq \lambda \leq k$, the set $Y_{\{j_1, \dots, j_k\}}$ can be taken as follows,

$$Y_{\{j_1, \dots, j_k\}} = \bigcup_{\lambda=1}^k \{x \phi_{j_\lambda, j_1 \cdots j_k} \mid x \in X_{j_\lambda}\}$$

For each $1 \leq \lambda \leq k$, the set X_{j_λ} is finite, since the union of finitely many finite sets is finite, and thus the set $Y_{\{j_1, \dots, j_k\}}$ is a finite subset of $S_{j_1 \cdots j_k}$. Since $S_{j_1 \cdots j_k}$ is locally finite, then $\langle Y_{\{j_1, \dots, j_k\}} \rangle$ is finite. As J has subsets with finitely many numbers, the set

$$T = \bigcup_{\emptyset \neq \{j_1, \dots, j_k\} \subseteq J} \langle Y_{\{j_1, \dots, j_k\}} \rangle$$

is finite.

Now, it is shown that the set $\langle X \rangle$ is finite. For any $s \in \langle X \rangle$, there exists $x_1 \in X_{j_1}, \dots, x_m \in X_{j_m}$ such that $s = x_1 x_2 \cdots x_m$.

Let $K = \{j_1, j_2, \dots, j_m\}$, then $\emptyset \neq K \subseteq J$ and $s \in \langle Y_K \rangle \subseteq T$. So, $\langle X \rangle \subseteq T$ and $\langle X \rangle$ is finite. Hence $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is locally finite.

5. Locally Finite Presentability

Let S be a semigroup. If every finitely generated subsemigroup of S is finitely presented, then S is called *locally finitely presented*.

Theorem 5.1 Let I be a finite semilattice, and let S_i ($i \in I$) be a family of disjoint semigroups. The strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is locally finitely presented if and only if, for each $i \in I$, S_i is locally finitely presented.

Proof.

(\Rightarrow) Since each S_i ($i \in I$) is a subsemigroup of the strong semilattice S and the subsemigroups of the locally finitely presented semigroups are locally finitely presented, each semigroup S_i ($i \in I$) is locally finitely presented.

(\Leftarrow) Let each S_i ($i \in I$) be locally finitely presented. Let X be a finite subset of the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. Let define the set J as follows,

$$J = \{i \in I \mid X \cap S_i \neq \emptyset\}$$

Also, let define $p(K) \in I$ for all $\emptyset \neq K \subseteq J$ as

$$K = \{i_1, \dots, i_m\} \Rightarrow p(K) = i_1 \cdots i_m$$

Then, since J is finite, the set $P(J) \subseteq I$ that is defined by

$$P(J) = \{p(K) \mid \emptyset \neq K \subseteq J\}$$

is finite. It is obvious that $P(J)$ is a subsemigroup of I generated by J . So $P(J)$ is a semilattice.

Let $X_i = X \cap S_i$ be a set for all $i \in J$. Let K be a subset of J . Let define Y_K as follows

$$Y_K = \bigcup_{i \in K} \{x \phi_{i, p(K)} \mid x \in X_i\}$$

And for all $k \in P(K)$, Z_k can be defined as follows

$$Z_k = \bigcup_{\emptyset \neq K \subseteq J \text{ ve } p(K)=k} Y_K$$

Then $Z_k \subseteq S_k$ for all $k \in P(K)$. Since S_k is locally finitely presented, then $T_k = \langle Z_k \rangle$ the subsemigroup generated by Z_k is finitely presented.

Now it is shown that the subsemigroup $\langle X \rangle$ is finitely presented. For any $s \in \langle X \rangle$, there exists

$$x_1 \in X_{j_1}, \dots, x_m \in X_{j_m} \text{ such that } s = x_1 x_2 \cdots x_m.$$

Let $K_s = \{j_1, j_2, \dots, j_m\}$. If $p(K_s) = k_s$, then $s \in T_{k_s}$.

Also from the definition of T_k , $T_k \subseteq \langle X \rangle$. Since each S_k is disjoint, then each T_k is a disjoint subsemigroup of $\langle X \rangle$ and $P(J)$ is finite, and so $\langle X \rangle$ is the union of the finitely many finitely presented disjoint semigroups T_k , that is,

$$\langle X \rangle = \bigcup_{k \in P(J)} T_k$$

Finally, let $\phi_{k,i}$ be the restriction of the homomorphisms of $\phi_{k,i}$ to the subsemigroup $T_k = \langle Z_k \rangle$. So it is clear that the strong semilattice $\mathcal{S}[P(J); T_k, \phi_{k,i}]$. Hence from Theorem 1 in [4], $\langle X \rangle$ is finitely presented.

6. Residual Finiteness

Residual finiteness is one of the most important algebraic finiteness conditions, and has been widely studied in the context of groups, semigroups, monoids and other algebraic structures. Let X be a set and let π be an equivalence relation

on X . For $x \in X$ we use x/π to denote the equivalence class of the element x , and X/π to denote the set of all equivalence classes. The number of π -classes of X is denoted $[X: \pi]$ and called this the *index* of the relation π . It is said that π separates the elements s and t if $s/\pi = t/\pi$. A semigroup S is *residually finite* if for every two distinct $x, y \in S$ there is a congruence ρ on S which has finite index and which separates x and y . This is equivalent to saying that there is a homomorphism Φ from S onto a finite semigroup such that $\Phi(s) \neq \Phi(t)$.

Theorem 6.1

Let I be a finite semilattice, and let S_i ($i \in I$) be a family of disjoint semigroups. The strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is residually finite if and only if, for each $i \in I$, S_i is residually finite.

Proof.

(\Rightarrow) Since each S_i ($i \in I$) is a subsemigroup of the strong semilattice S and the subsemigroups of the residually finite semigroups are residually finite, each semigroup S_i ($i \in I$) is residually finite.

(\Leftarrow) Suppose that each semigroup S_i ($i \in I$) is residually finite. Then, for all $i \in I$, there exist homomorphisms σ_i from S_i onto a finite semigroup S'_i such that $\sigma_i(w_{i_k}) \neq \sigma_i(w_{i_l})$ for each $w_{i_k}, w_{i_l} \in S_i$.

Let w_{i_k} and w_{i_l} be two different elements in the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. If w_{i_k} and w_{i_l} are both in the same subsemigroup S_i of S , then there exists a finite semigroup $S' = \bigcup_{i \in I} S'_i$ and a homomorphism $\varphi(w_{i_k}) = \sigma_i(w_{i_k})$. It is clear that φ is well-defined onto homomorphism from S to S' such that $\varphi(w_{i_k}) = \sigma_i(w_{i_k}) \neq \sigma_i(w_{i_l}) = \varphi(w_{i_l})$.

If they are not in the same subsemigroup of the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$. Then both of them are distinct elements. Assume that $w_{i_n} \in S_i$ and $w_{j_m} \in S_j$. Since S_i and S_j are residually finite, there are finite semigroups S'_i, S'_j and homomorphisms $\sigma_i: S_i \rightarrow S'_i$, $\sigma_j: S_j \rightarrow S'_j$.

If the mapping ψ is defined from the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ to the finite semigroup $S' = \bigcup_{i \in I} S'_i$ as follows,

$$\psi(w_{i_n}) = \begin{cases} \sigma_i(w_{i_n}), & w_{i_n} \in S_i \ (i \in I) \\ \sigma_j(w_{i_n}), & w_{i_n} \in S_j \ (j \in I) \end{cases}$$

Since σ_i is onto, it is clear that ψ is well-defined. Thus it follows that $\psi: S \rightarrow S'$ is an onto homomorphism such that $\psi(w_{i_n}) \neq \psi(w_{i_m})$. Thus, the strong semilattice $S = \mathcal{S}[I; S_i, \phi_{j,i}]$ is residually finite.

7. Conclusion

One of semigroup structures is a strong semilattice of semigroups which is an important structure for completely regular semigroups.

In this article it is proven that a strong semilattice of semigroups $\mathcal{S}[I; S_i, \phi_{j,i}]$ is periodic, local finite, locally finitely presented and residually finite, respectively if and only if I is finite and each semigroup S_i ($i \in I$) is periodic, locally finite, locally finitely presented and residually finite, respectively.

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