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Pentacyclic Harmonic Graph

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Abstract: Let G be a graph on n vertices $v_1, v_2, ..., v_n$ and let $d(v_i)$ be the degree of vertex v_i. A graph G is defined to be harmonic if $(d(v_1), d(v_2), ..., d(v_n))^t$ is an eigenvector of the $(0,1)$-adjacency matrix of G. We now show that there are 4 regular and 45 non-regular connected pentacyclic harmonic graphs and determine their structure. In the end we conclude that all of c-cyclic harmonic graphs for $1 \le c \le 5$ are planar graphs.

Keywords: Harmonic Graph, Eigenvalue, Spectra

1. Introduction

Let $G = (V(G), E(G))$ be a graph with $|V(G)| = n$ vertices $v_1, v_2, ..., v_n$ and $|E(G)| = m$ edges. We say that G is c-cyclic, whenever $c = m - n + p$, which p is the number components of G. In [1] B. Borovicanin and et al, studied the c-cycle graphs for $c = 1, 2, 3, 4$. All harmonic trees were constructed in [8] and the number of walks counted on some harmonic graph in [4, 5]. In [9, 10, 11] founded some result on harmonic graphs.

In this paper we study the c-cycle graphs for $c = 5$. If the graph G is connected and $c = 0$ then G is a tree.

The following elementary properties of harmonic graphs obtain of the spectra properties of graphs [2, 3, 6, 7]. Let $d(v_i)$ be the degree of vertex v_i for $1 \le i \le n$, that is the number of the first neighbors of v_i. Vertex of degree k is called a k-vertex. Vertex of degree zero is called pendent. The column-vector $(d(v_1), d(v_2), ..., d(v_n))^t$ is denoted by $d(G)$. The number of k-vertex denoted by n_k and we have

$$n_5 = 1 \tag{1}$$

$$n_3 = 0 \tag{2}$$

Definition 1. The adjacency matrix $A(G) = [a_{ij}]$ is the $n \times n$ matrix for which $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. Eigenvalues and eigenvectors of matrix $A(G)$ is called eigenvalues and eigenvectors of graph G.

Definition 2. A graph G is said to be harmonic if there exists a constant λ, such that

$$n_1 = 3,\ n_2 = 0,\ n_4 = 3 \tag{3}$$

In other words

$$A(G)d(G) = \lambda d(G). \tag{4}$$

Thus, graph G is harmonic if and only if $d(G)$ is one of its eigenvectors, theses graphs are called λ-harmonic. Equation (3) result that λ is a rational number and equation (4) implies that λ is not proper fraction, they follows that λ must be an integer.

Example 1. A λ-regular graph is a λ-harmonic graph.

By summing the expressions (3) over all $i = 1, 2, ..., n$ we have

$$\sum_{v \in V(G)} d(v)(d(v) - \lambda) = 0 \tag{5}$$

equivalently

$$\sum_{k \ge 0} k(k - \lambda)n_k = 0. \tag{6}$$

2. Some Auxiliary Results

We have the follow results of [1].

Lemma 1.

i. Let H be a graph obtained from G by adding to it an arbitrary number of isolated vertices, then H is harmonic if and only if G is harmonic.

ii. Any graph without isolated vertices is λ-harmonic if and only if all its components are λ-harmonic.

iii. Let G be a connected λ-harmonic graph. Then λ is greatest eigenvalue of G and its multiplicity is one. Also if $m>0$ then $\lambda \geq 1$ and equality occurs if and only if $G=K_2$.

From Lemma 2.1., it is enough to restrict our considerations to connected non-regular graphs. In [8], shown that for any positive integer λ there is a unique connected λ-harmonic that is a tree and denoted by T_λ. T_λ has $\lambda^3-\lambda^2+\lambda+1$ vertices, of which one vertices is a $(\lambda^2-\lambda+1)$-vertex, $(\lambda^2-\lambda+1)$ vertices are λ-vertices and $(\lambda-1)(\lambda^2-\lambda+1)$ vertices are pendant. Also in [1], shown that the following lemmas:

Lemma 2.

a In a λ-harmonic graph 1-vertex is adjacent to a vertex of degree λ.

b If a λ-harmonic graph not regular, then it has a vertex of degree greater than λ.

c In a harmonic graph with $n>2$, no 1-vertex is attached to any vertex of greatest degree.

Lemma 3.

The tree T_2 is the unique connected non-regular 2-harmonic graph.

Lemma 4.

If x is a vertex of a λ-harmonic graph then $d(x)\leq \lambda^2-\lambda+1$. If $d(x)=\lambda^2-\lambda+1$ then x belongs to a tree T_λ, otherwise $d(x)<\lambda^2-\lambda+1$.

Lemma 5: For the λ-harmonic tree, $n_1=(\lambda-1)n_\lambda$. For any other connected λ-harmonic graph, $n_1\leq(\lambda-2)n_\lambda$.

Lemma 6.

If $G\neq T_\lambda$ be a connected c-cyclic λ-harmonic graph with $\lambda\geq 3$, then $c\geq \frac{1}{2}(\lambda^2-2\lambda+2)$.

Lemma 7.

Let v be a vertex of a λ-harmonic graph such that $d(v)>\lambda^2-3\lambda+4$, and let u be a vertex adjacent to v, then $d(u)=\lambda$.

After then, we suppose that c-cyclic graphs are connected, that is $p=1$ therefore $m=n+c-1$. By combining the equalities (1) and (2) we get

$$\sum_{k\geq 0} k(k-2)n_k = 2c-2. \tag{7}$$

3. The Main Result

Theorem 1.

There are exactly 45 non-regular connected pentacyclic harmonic graphs, depicted in Figures 1- 17.

Proof:

Because of Lemma 6, if $c=5$ then λ cannot be greater than 4. Since $c=5$, therefore, $m=n+4$, on the other hand, Lemmas 2 and 5, result that λ cannot equal to1 and 2, hence $\lambda=3$ or $\lambda=4$. At the first, suppose that $\lambda=3$.

By the Lemma 2.4 if Δ is the maximal degree in a pentacyclic harmonic graph, then $\Delta\leq 6$ and in case $\lambda=4$ by Lemma 4 we have $\Delta\leq 12$. From Lemma 2 we the conclude that only the following 11 cases need to be examined:

Case 1: $\lambda=3,\quad \Delta=6$

Case 2: $\lambda=3,\quad \Delta=5$

Case 3: $\lambda=3,\quad \Delta=4$

Case 4: $\lambda=4,\quad \Delta=12$

Case 5: $\lambda=4,\quad \Delta=11$

Case 6: $\lambda=4,\quad \Delta=10$

Case 7: $\lambda=4,\quad \Delta=9$

Case 8: $\lambda=4,\quad \Delta=8$

Case 9: $\lambda=4,\quad \Delta=7$

Case 10: $\lambda=4,\quad \Delta=6$

Case 11: $\lambda=4,\quad \Delta=5$

Case 1: Lemma 5 implies that $n_3-n_1\geq 0$. By means of relation (7), for $c=5$, we have

$$-n_1+n_3+2n_4+3n_5+4n_6=8 \tag{8}$$

from which

$$2n_4+3n_5+4n_6-8=n_1-n_3\leq 0 \tag{9}$$

and we can conclude that

$$1\leq n_6\leq 2,\ n_5\leq 1,\ n_4\leq 2 \tag{10}$$

from equation (1.6) we get

$$-2n_1-2n_2+4n_4+10n_5+18n_6=0. \tag{11}$$

According to Lemma 7, the 5 and 6-vertices are adjacent only to 3-vertices. Since (3) the two neighbors of every 3-vertex, adjacent to a 6-vertex, must be a 1 and 2-vertex. Therefore $n_1\geq 6$, $n_2\geq 3$, and consequently, $n_1+n_2\geq 9$. In what follows we distinguish between 12 subcases:

Subcase 1:

$$n_4=0,\ n_5=0,\ n_6=1,\ n_1+n_2=9,\ n_3=n_1+4 \tag{12}$$

In this subcase, we have, $n_1=6$, $n_2=3$, $n_3=10$, $n_4=0$, $n_5=0$, $n_6=1$. Each of the three 2-vertices must be adjacent to two 3-vertices, and exacts of 4, 3-vertices remains. Therefore there cannot exist a 3-harmonic satisfies the condition (12).

Subcase 2:

$$n_4 = 1,\ n_5 = 0,\ n_6 = 1,\ n_1 + n_2 = 11,\ n_3 = n_1 + 2 \quad (13)$$

The 4 and 6-vertices are adjacent only to 3-vertices and therefore the number of 3-vertices is greater than or equal to 10. Because of $n_2 \geq 3$ we now have $n_1 = n_3 - 2 \geq 8$. Then, in this subcase we get

$$n_1 = 8,\ n_2 = 3,\ n_3 = 10,\ n_4 = 1,\ n_5 = 0,\ n_6 = 1 \quad (14)$$

the neighbors 3-vertex adjacent to 4-vertex has a 2-vertex, a 3-vertex, so we need at least 5, 2-vertices. This subcase also impossible.

Subcase 3:

$$n_4 = 2,\ n_5 = 0,\ n_6 = 1,\ n_1 + n_2 = 13,\ n_3 = n_1 \quad (15)$$

Similar arguments subcase 2, we have

$$n_1 = 10,\ n_2 = 3,\ n_3 = 10,\ n_4 = 2,\ n_5 = 0,\ n_6 = 1 \quad (16)$$

this graph is nonconnected, then this subcase is impossible.

Subcase 4:

$$n_4 = 0,\ n_5 = 1,\ n_6 = 1,\ n_1 + n_2 = 14,\ n_3 = n_1 + 1 \quad (17)$$

The 5 and 6-vertices are adjacent only to 3-vertices and therefore the number of 3-vertices is greater than or equal to 11. Because of $n_2 \geq 3$ we now have $n_1 = n_3 - 1 \geq 10$. Then, in this subcase we get

Table 1. *Cases of* $n_4 = 0, n_5 = 1, n_6 = 1, n_1 + n_2 = 14, n_3 = n_1 + 1$.

	n_1	n_2	n_3	n_4	n_5	n_6
(a)	10	4	11	0	1	1
(b)	11	3	12	0	1	1

the only case (a) can occurs and its graph is as follows.

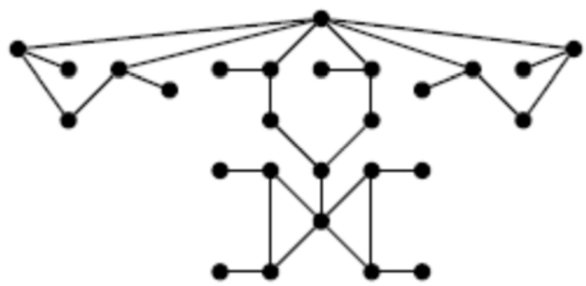

Figure 1. *First member of a family of 3-harmonic graphs which c=5 and* $\Delta = 6$.

Subcase 5:

$$n_4 = 1,\ n_5 = 1,\ n_6 = 1,\ n_1 + n_2 = 16,\ n_3 = n_1 - 1 \quad (18)$$

Since 3-vertices adjacent to 4-vertex, 5-vertex, 6-vertex, are distinct, so we need at least 15, 3-vertices, then $n_1 \geq 16$. It result that $n_1 + n_2 \geq 19$ that impossible.

Subcase 6:

$$n_4 = 2,\ n_5 = 1,\ n_6 = 1,\ n_1 + n_2 = 18,\ n_3 = n_1 - 3 \quad (19)$$

Similar arguments Subcase 1.5 this subcase is impossible.

Subcase 7:

$$n_4 = 0,\ n_5 = 0,\ n_6 = 2,\ n_1 + n_2 = 16,\ n_3 = n_1 \quad (20)$$

Since 3-vertices adjacent to 6-vertices are distinct, so we need at least 12, 3-vertices and 6, 2-vertices, then $n_1 \geq 12$. It result that $n_1 + n_2 = n_3 + n_2 \geq 18$ that impossible.

Subcase 8:

$$n_4 = 1,\ n_5 = 0,\ n_6 = 2,\ n_1 + n_2 = 20,\ n_3 = n_1 - 2 \quad (21)$$

In this subcase we have $n_3 \geq 16$ then $n_1 \geq 18$ and $n_1 + n_2 \geq 21$ that impossible.

Subcase 9:

$$n_4 = 2,\ n_5 = 0,\ n_6 = 2,\ n_1 + n_2 = 22,\ n_3 = n_1 - 4 \quad (22)$$

In this subcase we have $n_3 \geq 16$ then $n_1 \geq 20$ and $n_1 + n_2 \geq 23$ that impossible.

Subcase 10:

$$n_4 = 0,\ n_5 = 1,\ n_6 = 2,\ n_1 + n_2 = 23,\ n_3 = n_1 - 3 \quad (23)$$

In this subcase we have $n_3 \geq 17$ therefore $n_1 \geq 20$ and $n_1 + n_2 \geq 26$ that this is a contradiction.

Subcase 11:

$$n_4 = 1,\ n_5 = 1,\ n_6 = 2,\ n_1 + n_2 = 25,\ n_3 = n_1 - 5 \quad (24)$$

In this subcase we have $n_3 \geq 21$ thus $n_1 \geq 26$ and $n_1 + n_2 \geq 32$ that this cannot happen.

Subcase 12:

$$n_4 = 2,\ n_5 = 1,\ n_6 = 2,\ n_1 + n_2 = 27,\ n_3 = n_1 - 7 \quad (25)$$

In this subcase we get $n_3 \geq 21$ hence $n_1 \geq 28$ and $n_1 + n_2 \geq 34$ that impossible.

Case 2: $\lambda = 3, \Delta = 5$

Lemma 5 follows that $n_3 - n_1 \geq 0$. Equations (6) and (7) now became

$$-2n_1 - 2n_2 + 4n_4 + 10n_5 + 4n_6 = 0, \quad (26)$$

and

$$-n_1 + n_3 + 2n_4 + 3n_5 = 8. \quad (27)$$

We have following subcases:

Subcase 13:

$$n_4 = 0,\ n_5 = 2,\ n_1 + n_2 = 10,\ n_3 = n_1 + 2 \quad (28)$$

By the Lemma 7, every 5-vertex is adjacent only with 3-

vertices. Therefore $n_3 \geq 10$ and $n_1 \geq 8$, then we have

Table 2. *Cases of* $n_4 = 0, n_5 = 2, n_1 + n_2 = 10, n_3 = n_1 + 2$.

	n_1	n_2	n_3	n_4	n_5
(a)	8	2	10	0	2
(b)	9	1	11	0	2
(c)	10	0	12	0	2

Case (b) does not hold, but for case (a) we have 3 harmonic graphs as follow:

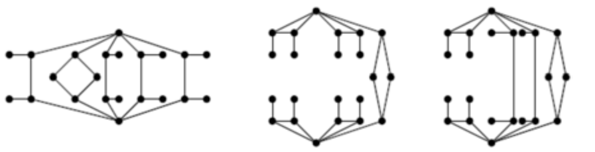

Figure 2. *First members of a family of 3-harmonic graphs which c=5 and* $\Delta = 5$.

and for case (c) we have 8 harmonic graphs as follow:

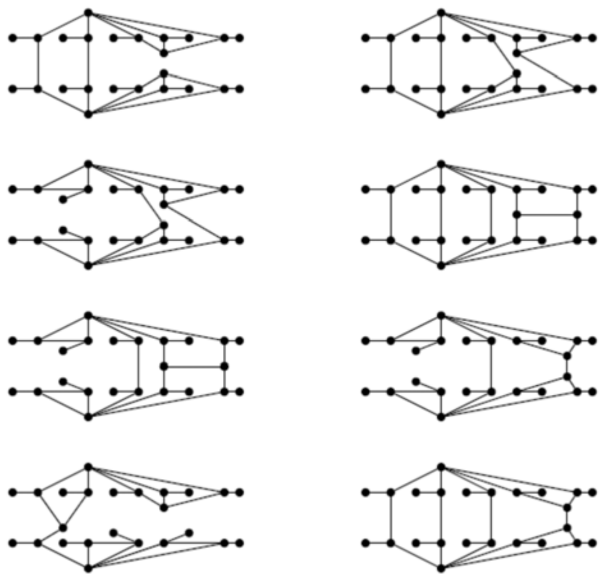

Figure 3. *Second members of a family of 3-harmonic graphs which c=5 and* $\Delta = 5$.

Subcase 14:

$$n_4 = 1, n_5 = 2, n_1 + n_2 = 12, n_3 = n_1 \tag{29}$$

In this subcase we have $n_3 \geq 14$ then $n_1 \geq 14$ and $n_1 + n_2 \geq 14$ that impossible.

Subcase 15:

$$n_4 = 2, n_5 = 2, n_1 + n_2 = 14, n_3 = n_1 - 2 \tag{30}$$

Since $n_3 - n_1 \geq 0$ then this subcase is impossible.

Subcase 16:

$$n_4 = 0, n_5 = 1, n_1 + n_2 = 5, n_3 = n_1 + 5 \tag{31}$$

In this subcase we have $n_3 \geq 5$ then we have

Table 3. *Cases of* $n_4 = 0, n_5 = 1, n_1 + n_2 = 5, n_3 = n_1 + 5$.

	n_1	n_2	n_3	n_4	n_5
(a)	0	5	5	0	1
(b)	1	4	6	0	1
(c)	2	3	7	0	1
(d)	3	2	8	0	1
(e)	4	1	9	0	1
(f)	5	0	10	0	1

cases (b), (c) and (e) do not hold, but for case (a) we have 2 harmonic graphs as follow:

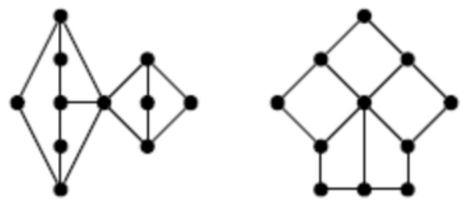

Figure 4. *Third members of a family of 3-harmonic graphs which c=5 and* $\Delta = 5$.

for case (d) we have 1 harmonic graphs as follow:

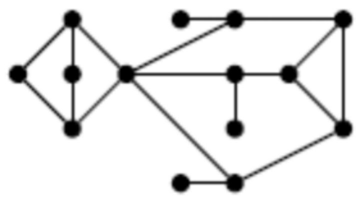

Figure 5. *Fourth members of a family of 3-harmonic graphs which c=5 and* $\Delta = 5$.

for case (f) we have 3 harmonic graphs as follow:

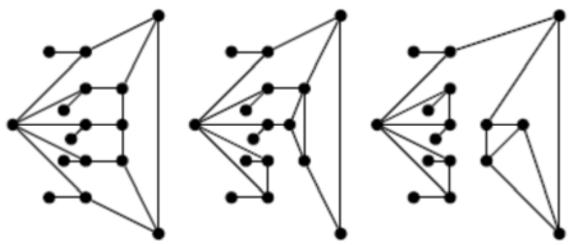

Figure 6. *Fifth members of a family of 3-harmonic graphs which c=5 and* $\Delta = 5$.

Subcase 17:

$$n_4 = 1, n_5 = 1, n_1 + n_2 = 7, n_3 = n_1 + 3 \tag{32}$$

In this subcase we have $n_3 \geq 9$ then $n_1 \geq 6$ and

Table 4. *Cases of* $n_4 = 1, n_5 = 1, n_1 + n_2 = 7, n_3 = n_1 + 3$.

	n_1	n_2	n_3	n_4	n_5
(a)	6	1	9	1	1
(b)	7	0	10	1	1

since the neighbors 3-vertex adjacent to 4-vertex has a 2-vertex, a 3-vertex, so we need at least 2, 2-vertices, then this subcase is impossible.

Subcase 18:

$$n_4 = 2,\ n_5 = 1,\ n_1 + n_2 = 9,\ n_3 = n_1 + 1 \qquad (33)$$

In this subcase we have $n_3 \geq 9$ hence $n_1 \geq 8$ and

Table 5. *Cases of* $n_4 = 2, n_5 = 1, n_1 + n_2 = 9, n_3 = n_1 + 1$.

	n_1	n_2	n_3	n_4	n_5
(a)	8	1	9	2	1
(b)	9	0	10	2	1

since some the neighbors 3-vertex adjacent to 5-vertex has a 2-vertex, a 3-vertex, so we need at least 2, 2-vertices, then this subcase is impossible.

Case 3: $\lambda = 3, \Delta = 4$

Lemma 5 follows that $n_3 - n_1 \geq 0$. Equations (6) and (7) now became

$$-2n_1 - 2n_2 + 4n_4 = 0, \qquad (34)$$

and

$$-n_1 + n_3 + 2n_4 = 8 \qquad (35)$$

$n_3 - n_1 \geq 0$ results that $1 \leq n_4 \leq 4$ and the other hand $n_1 + n_2 = 2n_4$, then we have following subcases:

Subcase 19:

$$n_4 = 1,\ n_1 + n_2 = 2,\ n_3 = n_1 + 6 \qquad (36)$$

Equivalently

Table 6. *Cases of* $n_4 = 1, n_1 + n_2 = 2, n_3 = n_1 + 6$.

	n_1	n_2	n_3	n_4
(a)	1	1	7	1
(b)	2	0	8	1
(c)	0	2	6	1

since the neighbors 3-vertex adjacent to 4-vertex has a 2-vertex, a 3-vertex, so we need at least 2, 2-vertices, then subcases (a) and (b) are impossible, but for case (c) we have 2 harmonic graphs as follow:

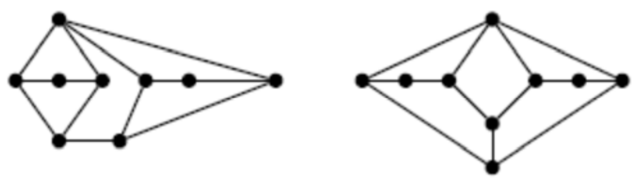

Figure7. *First members of a family of 3-harmonic graphs which c=5 and* $\Delta = 4$.

Subcase 20:

$$n_4 = 2,\ n_1 + n_2 = 4,\ n_3 = n_1 + 4 \qquad (37)$$

equivalently

Table 7. *Cases of* $n_4 = 2, n_1 + n_2 = 4, n_3 = n_1 + 4$.

	n_1	n_2	n_3	n_4
(a)	0	4	4	2
(b)	1	3	5	2
(c)	2	2	6	2
(d)	3	1	7	2
(e)	4	0	8	2

for case (a) we have 3 harmonic graphs as follow:

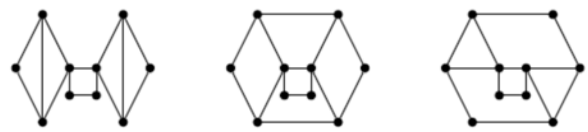

Figure 8. *Second members of a family of 3-harmonic graphs which c=5 and* $\Delta = 4$.

In subcase (b) we need at least 6, 3-vertices, then this subcase is impossible, also subcases (d) and (e) are impossible, but for case (c) we have 3 harmonic graphs as follow:

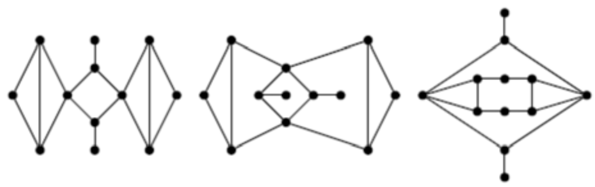

Figure 9. *Third members of a family of 3-harmonic graphs which c=5 and* $\Delta = 4$.

Subcase 21:

$$n_4 = 3,\ n_1 + n_2 = 6,\ n_3 = n_1 + 2 \qquad (38)$$

If any two 4-vertices of three 4-vertices be adjacent, then $n_3 = 0$, thus this manner cannot occurs. If just a 4-vertex adjacent with the other 4-vertices, then we have 2 harmonic graphs as follow:

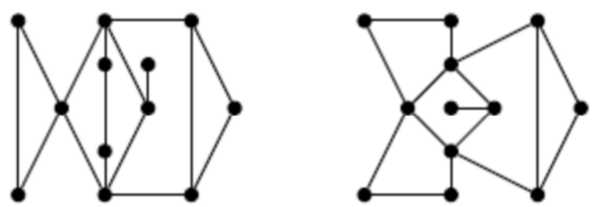

Figure 10. *Fourth members of a family of 3-harmonic graphs which c=5 and* $\Delta = 4$.

if the only two 4-vertices of three 4-vertices be adjacent, then we have 2 harmonic graphs as follow:

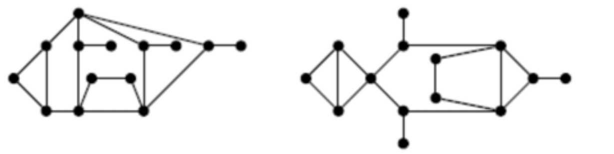

Figure 11. Fifth members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

Subcase 22:

$$n_4 = 4,\ n_1 + n_2 = 8,\ n_3 = n_1 \tag{39}$$

equivalently

Table 8. *Cases of* $n_4 = 4,\ n_1 + n_2 = 8,\ n_3 = n_1$.

	n_1	n_2	n_3	n_4
(a)	0	8	0	4
(b)	1	7	1	4
(c)	2	6	2	4
(d)	3	5	3	4
(e)	4	4	4	4
(f)	5	3	5	4
(g)	6	2	6	4
(h)	7	1	7	4
(k)	8	0	8	4

In above subcases we need to the even number of 3-vertices, then these subcases (b), (d), (e) and (h) are impossible. For case (a) we have 4 harmonic graphs as follow:

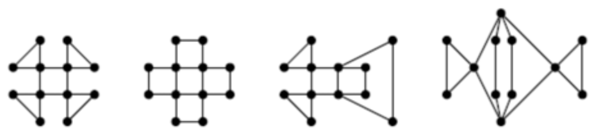

Figure 12. Sixth members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

for case (c) we have 3 harmonic graphs as follow:

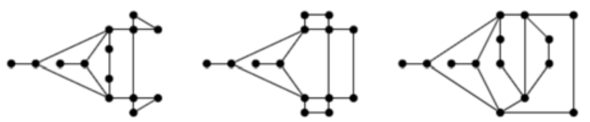

Figure 13. Seventh members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

for case (e) we have 2 harmonic graphs as follow:

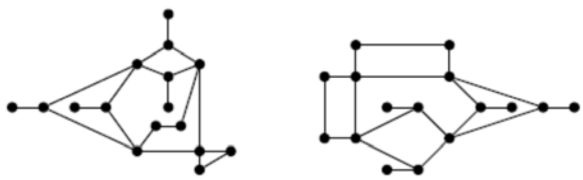

Figure 14. Eighth members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

for case (g) we have 2 harmonic graphs as follow:

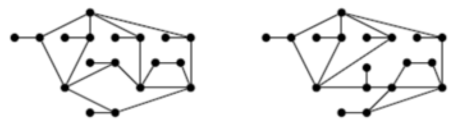

Figure 15. Ninth members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

and also for case (k) we have 3 harmonic graphs as follow:

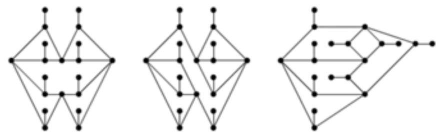

Figure 16. Tenth members of a family of 3-harmonic graphs which c=5 and $\Delta = 4$.

Case 4: $\lambda = 4,\ \Delta = 12$

Lemma 5 follows that $2n_4 - n_1 \ge 0$. Equations (6) and (7) now became

$$\begin{aligned}&-3n_1 - 4n_2 - 3n_3 + 5n_5 + 12n_6 + 21n_7 + 32n_8\\&+45n_9 + 60n_{10} + 77n_{11} + 96n_{12} = 0,\end{aligned} \tag{40}$$

and

$$\begin{aligned}&-n_1 + n_3 + 2n_4 + 3n_5 + 4n_6 + 5n_7 + 6n_8\\&+7n_9 + 8n_{10} + 9n_{11} + 10n_{12} = 8.\end{aligned} \tag{41}$$

Since $n_{12} \ge 1$ and $2n_4 - n_1 \ge 0$ then this case is impossible.

Case 5: $\lambda = 4,\ \Delta = 11$

Lemma 5 follows that $2n_4 - n_1 \ge 0$. Equations (6) and (7) now became

$$\begin{aligned}&-3n_1 - 4n_2 - 3n_3 + 5n_5 + 12n_6 + 21n_7\\&+32n_8 + 45n_9 + 60n_{10} + 77n_{11} = 0,\end{aligned} \tag{42}$$

and

$$\begin{aligned}&-n_1 + n_3 + 2n_4 + 3n_5 + 4n_6 + 5n_7\\&+6n_8 + 7n_9 + 8n_{10} + 9n_{11} = 8.\end{aligned} \tag{43}$$

Because of $n_{11} \ge 1$ and $2n_4 - n_1 \ge 0$ this cannot happen.

Case 6: $\boldsymbol{\lambda = 4,\ \Delta = 10}$

Lemma 5 follows that $2n_4 - n_1 \ge 0$. Equations (6) and (7) now became

$$\begin{aligned}&-3n_1 - 4n_2 - 3n_3 + 5n_5 + 12n_6\\&+21n_7 + 32n_8 + 45n_9 + 60n_{10} = 0,\end{aligned} \tag{44}$$

and

$$\begin{aligned}&-n_1 + n_3 + 2n_4 + 3n_5 + 4n_6\\&+5n_7 + 6n_8 + 7n_9 + 8n_{10} = 8.\end{aligned} \tag{45}$$

Since $n_{10} \ge 1$ and $2n_4 - n_1 \ge 0$ then (45) implies that

$$n_{10}=1,\ 2n_4=n_1,\\ n_3=n_5=n_6=n_7=n_8=n_9=0 \tag{46}$$

and (45) results that $3n_1+4n_2=60$. Since vertices adjacent to 10-vertex are 4-vertex then $n_4\ge 10$ and $n_1\ge 20$. The equation $3n_1+4n_2=60$ leads to $n_1=20, n_2=0, n_4=10$. This harmonic graph as follow:

Figure 17. *First member of a family of 4-harmonic graphs which c=5 and* $\Delta=10$.

Case 7: $\lambda=4, \Delta=9$

From Lemma 5 we get $2n_4-n_1\ge 0$. Equations (6) and (7) imply that

$$-3n_1-4n_2-3n_3+5n_5+12n_6\\ +21n_7+32n_8+45n_9=0, \tag{47}$$

and

$$-n_1+n_3+2n_4+3n_5+\\ 4n_6+5n_7+6n_8+7n_9=8. \tag{48}$$

Because of $n_9\ge 1$ and $2n_4-n_1\ge 0$ the relation (48) implies that

$$n_9=1,\ 2n_4=n_1,\ n_3=1,\ n_5=n_6=n_7=n_8=0 \tag{49}$$

or

$$n_9=1,\ 2n_4-1=n_1,\ n_3=n_5=n_6=n_7=n_8=0 \tag{50}$$

and (47) results that $3n_1+4n_2+3n_3=45$. Since vertices adjacent to 9-vertex are 4-vertex then $n_4\ge 9$ and $n_1\ge 17$. The equation $3n_1+4n_2+3n_3=45$ results that this case is impossible.

Case 8: $\lambda=4, \Delta=8$

Lemma 5 leads to $2n_4-n_1\ge 0$. From equations (6) and (7) we get

$$-3n_1-4n_2-3n_3+5n_5\\ +12n_6+21n_7+32n_8=0, \tag{51}$$

and

$$-n_1+n_3+2n_4+3n_5+4n_6+5n_7+6n_8=8. \tag{52}$$

Since $n_8\ge 1$ and $2n_4-n_1\ge 0$ then (52) implies that

$$n_8=1,\ 2n_4-n_1+n_3=2,\ n_5=n_6=n_7=0 \tag{53}$$

and (51) implies that $3n_1+4n_2+3n_3=32$. Since vertices adjacent to 8-vertex are 4-vertex so $n_4\ge 8$ and $n_1\ge 14$. The equation $3n_1+4n_2+3n_3=32$ results that this case is impossible.

Case 9: $\lambda=4, \Delta=7$

From Lemma 5 we have $2n_4-n_1\ge 0$. Equations (6) and (7) lead to

$$-3n_1-4n_2-3n_3+5n_5+12n_6+21n_7=0, \tag{54}$$

and

$$-n_1+n_3+2n_4+3n_5+4n_6+5n_7=8. \tag{55}$$

Since $n_7\ge 1$ and $2n_4-n_1\ge 0$ then (55) implies that

$$n_7=1,\ 2n_4-n_1+n_3+3n_5=3,\ n_6=0 \tag{56}$$

and (54) results that $3n_1+4n_2+3n_3-5n_5=21$. If $n_5=0$ then vertices adjacent to 7-vertex are 4-vertex then $n_4\ge 7$ and $n_1\ge 11$. The equation $3n_1+4n_2+3n_3=21$ results that this case is impossible. Also, if $n_5=1$ then $n_3=0$ and $2n_4=n_1$, then since some of vertices adjacent to 7-vertex are 4-vertex then $n_4\ge 5$ and $n_1\ge 10$. The equation $3n_1+4n_2=26$ results that this case is impossible.

Case 10: $\lambda=4, \Delta=6$

Lemma 5 follows that $2n_4-n_1\ge 0$. Equations (6) and (7) now became

$$-3n_1-4n_2-3n_3+5n_5+12n_6=0, \tag{57}$$

and

$$-n_1+n_3+2n_4+3n_5+4n_6=8. \tag{58}$$

We have following subcases:

Subcase 23:

$$n_6=1,\ 2n_4=n_1,\ n_3+3n_5=4,\\ -3n_1-4n_2-3n_3+5n_5+12=0 \tag{59}$$

If $n_5=0$ then $n_3=4$, $n_1=n_2=n_4=0$ therefore this manner not occurs. If $n_5=1$ then $n_3=1$ and $3n_1+4n_2=14$. On the other hand $n_4\ge 4$ thus $n_1\ge 8$ that this contradiction with $3n_1+4n_2=14$. Hence this manner also cannot occurs.

Subcase 24:

$$n_6=1,\ 2n_4=n_1+1,\ n_3+3n_5=3,\\ -3n_1-4n_2-3n_3+5n_5+12=0 \tag{60}$$

If $n_5=0$ then $n_3=3$, $n_1=1, n_2=0, n_4=1$ therefore this manner is impossible. If $n_5=1$ then $n_3=0$ and $3n_1+4n_2=17$. Also, $n_4\ge 4$ thus $n_1\ge 8$ that this contradiction with $3n_1+4n_2=17$. Therefore this case also cannot occurs.

Subcase 25:

$$n_6 = 1,\ 2n_4 = n_1 + 2,\ n_3 + 3n_5 = 2, \\ -3n_1 - 4n_2 - 3n_3 + 5n_5 + 12 = 0 \tag{61}$$

In this subcases $n_3 = 2, n_5 = 0$, $n_1 = 2, n_2 = 0, n_4 = 2$ then this case is impossible.

Subcase 26:

$$n_6 = 1,\ 2n_4 = n_1 + 3,\ n_3 + 3n_5 = 1, \\ -3n_1 - 4n_2 - 3n_3 + 5n_5 + 12 = 0 \tag{62}$$

In this subcases $n_3 = 1, n_5 = 0$, $n_1 = 3, n_2 = 0, n_4 = 3$ then this case cannot happen.

Subcase 27:

$$n_6 = 1,\ 2n_4 = n_1 + 4,\ n_3 + 3n_5 = 0, \\ -3n_1 - 4n_2 - 3n_3 + 5n_5 + 12 = 0 \tag{63}$$

In this subcases $n_3 = n_5 = 0$, $3n_1 + 4n_2 = 12$. Since vertices adjacent to 6-vertex are 4-vertex then $n_4 \geq 6$ and $n_1 \geq 8$. So this contradiction with $3n_1 + 4n_2 = 12$. Then this case is impossible.

Subcase 28:

$$n_6 = 2,\ 2n_4 = n_1,\ n_3 = n_5 = 0, \\ 3n_1 + 4n_2 = 24 \tag{64}$$

since some vertices adjacent to 6-vertex are 4-vertex then $n_4 \geq 4$ and $n_1 \geq 8$, $n_2 \geq 2$. That this contradiction with $3n_1 + 4n_2 = 24$, then this case is impossible.

Case 11: $\lambda = 4, \Delta = 5$

Lemma 5 follows that $2n_4 - n_1 \geq 0$. Equations (6) and (7) now became

$$-3n_1 - 4n_2 - 3n_3 + 5n_5 = 0, \tag{65}$$

and

$$-n_1 + n_3 + 2n_4 + 3n_5 = 8. \tag{66}$$

$3n_1 + 4n_2 = 5n_5 - 3n_3 \geq 0$ then we have following subcases:

Subcase 29:

$$n_5 = 1,\ n_3 = 0,\ 2n_4 = n_1 + 5,\ 3n_1 + 4n_2 = 5 \tag{67}$$

Since vertices adjacent to 5-vertex are 4-vertex then $n_4 = 5$ and $n_1 = 5$ that this contradiction with $3n_1 + 4n_2 = 5$. Hence this case is impossible.

Subcase 30:

$$n_5 = 1,\ n_3 = 1,\ 2n_4 = n_1 + 4,\ 3n_1 + 4n_2 = 2 \tag{68}$$

Since $n_1 \geq 0$ and $n_2 \geq 0$, hence this case cannot happen.

Subcase 31:

$$n_5 = 2,\ n_3 = 0,\ 2n_4 = n_1 + 2,\ 3n_1 + 4n_2 = 10 \tag{69}$$

Since some vertices adjacent to 5-vertex are 4-vertex then $n_4 \geq 3$ and $n_1 \geq 4$ that this contradiction with $3n_1 + 4n_2 = 10$. Therefore this case is impossible.

Subcase 32:

$$n_5 = 2,\ n_3 = 1,\ 2n_4 = n_1 + 1,\ 3n_1 + 4n_2 = 7 \tag{70}$$

Some vertices adjacent to 5-vertex are 4-vertex then $n_4 \geq 3$ and $n_1 \geq 5$ that this contradiction with $3n_1 + 4n_2 = 7$. Thus this case is impossible.

Subcase 33:

$$n_5 = 2,\ n_3 = 2,\ 2n_4 = n_1,\ 3n_1 + 4n_2 = 4 \tag{71}$$

Since some vertices adjacent to 5-vertex are 4-vertex then $n_4 \geq 3$ and $n_1 \geq 6$ that this contradiction with $3n_1 + 4n_2 = 4$. Hence this case is impossible.

Definition 3:

A graph is planar if it can be drawn in a plane without graph edges crossing.

Corollary 1:

All of c-cyclic nonregular harmonic graphs for $c \leq 5$ are planar graphs.

4. Regular Harmonic Graphs

If a pentacyclic λ -harmonic graph be regular then we have $n = \frac{8}{\lambda - 2}$ and $n \geq \lambda + 1$, therefore we have the only $\lambda = 3, n = 8$. In this case we have 4, 3-harmonic graphs as follow:

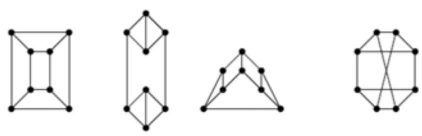

Figure 18. *Connected regular pentacyclic harmonic graphs.*

5. Conclusions

Let # r(c) and #nr(c) be denote the number of connected c-cyclic regular and nonregular harmonic graphs, respectively, for a fixed value c. According to fined results, the number of harmonic graphs as follows.

Table 9. *The number of harmonic graphs.*

c	# r(c)	#nr(c)	Remark
0	1	∞	$\lambda \geq 1$
1	∞	0	$\lambda = 2$
2	0	0	
3	1	4	$\lambda = 3$
4	2	18	$\lambda = 3$
5	4	45	$\lambda = 3,\ \lambda = 4$
≥ 6	finite	finite	

References

[1] Borovicanin B., Grunewald S., Gutman I., Petrovic M. 2003. Harmonic graphs with small number of cycles, Discrete Mathematics 265, 31-44.

[2] Cvetkovi'c D., Doob M., Sachs H. 1995. Spectra of Graphs: Theory and Applications (3rd ed.), Johann Ambrosius Barth, Heidelberg.

[3] Cvetkovi'c D., Rowlinson P., Simi'c S. 1997. Eigenspaces of Graphs, Cambridge University Press, Cambridge.

[4] Dress A., Gutman I. 2003. Asymptotic results regarding the number of walks in a graph, Appl. Math. Lett., 16(3), 389-393.

[5] Dress A., Gutman I. 2003. On the number of walks in a graph, Appl. Math. Lett. 16(5), 797-801.

[6] Favaron O., Mah'eo M., Sacl'e J.F., 1993. Some eigenvalue properties in graphs (conjectures of GraLti—II), Discrete Math. 111, 197–220.

[7] Godsil C., Royle G. 2001. Algebraic Graph Theory, Springer-Verlag, New York.

[8] Grunewald S. 2002. Harmonic tree, App. Math. Lett. 15, 1001-1004.

[9] Petrovi´c M., Borovi´canin B., Radosavljevi´c Z. 2006. The integral 3-harmonic graphs, Linear Algebra and its Applications 416, 298–312.

[10] Mahmoodi A. 2013. Some Results on Harmonic Graphs, Gen. Math. Notes, 19(1), 53-59.

[11] Salehi Zarrin Ghabaei A., Azami S. 2014. Some properties of harmonic graphs, Advances in Environmental Biology, 8(11), 597-605.