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Region Mathematics-a New Direction in Mathematics: Part-2

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Abstract: This is sequel to our earlier work [11] in which we introduced a new direction in Mathematics called by "Region Mathematics". The 'Region Mathematics' is a newly discovered mathematics to be viewed as a universal mathematics of super giant volume containing the existing rich volume of mathematics developed so far since the stone age of earth. To introduce the 'Region Mathematics', we began in [11] by introducing three of its initial giant family members: Region Algebra, Region Calculus and Multi-dimensional Region Calculus. In this paper we introduce three more new topics of Region Mathematics which are : Theory of Objects, Theory of A-numbers and Region Geometry. Several new kind of Numbers are discovered, and consequently the existing 'Theory of Numbers' needs to be updated, extended and viewed in a new style.

Keywords: Onteger, Prime Object, Imaginary Object, Complex Object, Compound Number, Region Geometry

1. Introduction

In the work [11] we introduced a new direction in 'Mathematics' called by "Region Mathematics" for the world mathematicians, academicians, scientists and engineers. The direction is launched by discovering the algebraic structure "Region" first of all, on submitting sufficient justification behind the genuine and mandatory need to introduce it; and then introducing the "Region Calculus". The new topic "Region Calculus" is a generalization of the classical calculus and Analysis [12, 19, 20, 22]. In this paper we introduce three new family members of "Region Mathematics", which are : "Theory of Objects", a new language of the "Theory of Numbers" (which generates the existing 'Theory of Numbers' as one particular instance of it), and finally another new topic "Region Geometry" which do also generate the classical geometry as one of its particular instance. But with the introduction of few of its giant family members, it is the beginning of the super-giant "Region Mathematics", and at this moment it is at baby stage. The purpose of developing the super-giant 'Region Mathematics' is not just for doing a kind of generalization of the existing rich volume of classical Mathematics, but it has automatically happened so in its initial growth in the work [11]. The complete content of the work [11] and of the present paper can be well studied without referring to the work [3-7], because the work [11] and also the present work is a major revised and updated version.

The properties of region algebra are very important as this is the 'minimal algebra' which justifies free and fluent practice of elementary as well as higher algebra. This important caliber of regions having the unique property to be qualified as the 'minimal algebra' in the sense of giving a kind of authorized driving license to the world mathematicians, the caliber which is not possessed by groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, and even not by 'division algebra' or by any existing standard algebraic system alone, in general, by virtue of their respective definitions and properties. This important identification, probably the most important issue in the subject 'Algebra' and one of the most important issues in Mathematics, was missing so far in any past literature of algebra [1, 2, 14-17, 21, 23], and thus it is surely a unique algebra of absolute integrated nature and super power. With the introduction of Region Mathematics, all existing branches of mathematics can be provided their siblings with the progress in future research works, in order to explore the academic universe of science, mathematics, engineering, social science, statistics, etc. with many more alternative new approaches and new thoughts. For details about Region Algebra and Region Calculus, in particular about an important notion called by 'Complete Region', one could see [11] as pre-requisites for the present work.

In section-2 here a new algebraic theory called by "Theory of Objects" is introduced which generates the existing very popular notion of 'prime/composite objects' and then produces the existing notion of prime/composite numbers as a special case of 'prime/composite objects'. The notion of 'imaginary objects' in a region is then introduced and it is observed that the classical 'imaginary numbers' (or, complex numbers) are just one particular instance of the 'imaginary objects'. Although the birth of the particular instance 'imaginary numbers' took place in an independent way long before (i.e. long before the discovery of 'imaginary objects'), but interestingly it happened out of a very particular 'region'!, the fact which is unearthed and explained here. Neither Division Algebra [1, 2, 14-17, 21, 23] nor any existing algebraic system alone can produce this theory on the development of prime numbers, composite numbers, imaginary numbers and compound numbers. One very interesting topic introduced is the discovery of 'compound numbers'.

We then introduce another new giant direction in Number Theory. We say that every complete region A has its own 'Theory of Numbers' called by 'Theory of A-numbers', where the classical 'Theory of Numbers' is just one instance of it being the the 'Theory of RR-numbers' corresponding to the particular complete region RR. It is claimed that the "Theory of Objects" will play a huge role to the Number Theorists in a new direction. In due time, the 'Number Theorists' may be re-designated with the new title 'Object Theorists' as they may need to cultivate the broad area 'Theory of Objects' in pursuance of cultivating the 'Theory of Numbers' in a much better style and fashion. In fact, one of the major contributions in this work on Region Mathematics is that several new type of numbers are discovered (in Section-2). All these new sets of numbers need to be studied further in the context of F-algebra, Associative Algebra and Division Algebra and ofcourse in the context of region algebra.

In Region Mathematics, the "Theory of Objects" then induces another new direction called by "Region Geometry" in Section-3. The "Region Geometry" is interesting, being a generalization of our rich classical geometry of the existing notion. The notion of object point, object axes, region line, region plane etc. are introduced here as the initial work on "Region Geometry".

2. Theory of Objects

'Region' is the most practiced algebra in school/college education, research, scientific and engineering calculations, etc. and a mandatory algebra in the study of science, mathematics, engineering. Its objects (elements) play various roles to expose themselves for induction in various branches of mathematics, and they exercise among themselves too with various characteristic properties. This phenomenon develops a new direction in Region Mathematics called by "Theory of Objects".

This section provides the beginning of the "Theory of Objects". Presently the theory is at its baby stage. In due time, with rigorous future research, it will include its other components too. However, the Theory is initiated in this section with three giant topics as follows:

1. "Prime Objects" and "Composite Objects" in a Region.
2. "Imaginary Objects" and "Compound Objects" in a Region.
3. "Theory of Numbers": Every Complete Region has its own.

The subsection 2.1 introduces the topic "Prime Objects" and "Composite Objects" in a Region, the subsection 2.2 introduces the topic "Imaginary Objects" and "Compound Objects" in a Region, and the subsection 2.3 introduces the topic "Theory of Numbers" : Every Complete Region has its own.

2.1. 'Prime Objects' and 'Composite Objects' in a Region

In our school mathematics, we speak about 'prime numbers' and 'composite numbers'. They are members of the set R of real numbers. We will see in this subsection that they are objects of the region R, viewing them as objects instead of numbers. In this sense we may view them as 'prime objects' and 'composite objects' of the region R. But there are few simple questions arise immediately: Why we are to consider the notion of 'prime objects' and 'composite objects' of the particular region R only? Why do we not think and explore about an analogous concept for other regions too? In this subsection we introduce the notion of 'prime object' and 'composite object' in any arbitrary region A = $(A, \oplus, *, \bullet)$. We consider here regions only, not necessarily complete regions. First of all we introduce the notion of 'bachelor set' in a given region.

2.1.1. 'Bachelor Set' in a Region

Let A be a region. A subset B of the region A is called a 'bachelor set' in A if

(i) $1_A \in B$, $0_A \notin B$ and
(ii) $\forall x (\neq 1_A) \in B$, $x^{-1} \notin B$.

Clearly, a bachelor set can never be a null set because the smallest bachelor set in a region A is the singleton $\{1_A\}$. Also, it is obvious from the above definition that the self-inverse objects (like an element x, where $x^2=1_A$) other than 1_A of the region A are not the members of any bachelor set of A.

Any subset S of a bachelor set B in the region A is also a bachelor set in A if $1_A \in S$. It can be verified that if B is a bachelor set in a region A, then the set $\tilde{B} = \{y : y = x^{-1}$ where $x \in B\}$ is also a bachelor set in A. This set $\tilde{B}$ is called the 'conjugate bachelor' of the bachelor set B in the region A.

Clearly, conjugate of the conjugate of B is B itself. The union of two bachelors in A need not be a bachelor in A, but the intersection of two bachelors will be a bachelor in A. For every bachelor set B in A, B $\cap \tilde{B}$ = $\{1_A\}$. If B and C are two bachelors in the region A, then the conjugate of (B$\cap$C) is $\tilde{B} \cap \tilde{C}$. If B = $\tilde{B}$, then the only case is that B = $\tilde{B}$ = $\{1_A\}$.

Example 2.1.1

Consider the region RR. Clearly the following are true :

(i) the set N of natural numbers is a bachelor set in the region RR.

(ii) The set M = {1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, …} = {m : m = 1/n, n ∈ N, where N is the set of natural numbers} is a bachelor set in the region RR.

(iii) The set L = {1, 78.261, 9287, 83.5} is also a bachelor set in the region RR.

Example 2.1.2

The set R^+ of all positive real numbers is not a bachelor set in the region RR.

Proposition 2.1.1

If the set B of cardinality n is a bachelor set in the region A, then B has 2^{n-1} number of distinct sub-bachelors.

Proof:

For n = 1, the result is true because the only possibility is that B = $\{1_A\}$.

Now consider the case n > 1. The two trivial sub-bachelors are $\{1_A\}$ and B. The cardinality of the set B – $\{1_A\}$ is (n-1) which is having 2^{n-1} number of subsets including the null set and the set B – $\{1_A\}$ itself. Adding the common element 1_A to each of these 2^{n-1} subsets will create 2^{n-1} number of bachelor sets of A, being all the sub-bachelors of B. Hence proved.

There are four types of division operations in region algebra which are defined in subsection 3.2.9 in our earlier work in [11]. We introduce here the operation of 'Exact Division' in a bachelor set in the region A, which is a kind of division of an element of a bachelor set B by another element of the same bachelor set B.

2.1.2. 'Exact Division' in a Bachelor Set

Let B be a bachelor set in the region A. Consider two objects x, y ∈ B. We say that the object x exactly divides the object y in B, denoted by the notation "x $|_B$y", if ∃ z ∈ B such that $\frac{y}{x}$ = z holds good in the region A. The notation "$|_B$" signifies the operation of 'exact division' in B, and the notation signifies the operation "can not exactly divide" in B.

The following results are straightforward.

Proposition 2.1.2

(i) x $|_B$ x and $1_A|_B$ x ∀ x ∈ B.

(ii) for x ≠ y, if x $|_B$ y then y x, where x, y ∈ B.

Proposition 2.1.3

It may happen that for a given pair of objects x, y in a bachelor B in a region A, neither x $|_B$ y nor y $|_B$ x.

Proof:

Consider a bachelor C in the region A where x, y are in C and x $|_C$ y (such that $\frac{y}{x}$ = z). Now consider the set B = C – {z}. Clearly B is a bachelor in the region A, where both x and y are in a bachelor B but neither x $|_B$ y nor y $|_B$ x. Hence proved.

2.1.3. 'Composite Objects' and 'Prime Objects' in a Region with Respect to a Bachelor Set of It

We introduce now the notion of 'Composite Objects' and 'Prime Objects' in a region with respect to a bachelor set B of it.

'Composite Object'

Let B be a bachelor set of a region A. An object x ∈ B is called a 'Composite Object' in B, if ∃ p, q ∈ B – $\{1_A\}$ such that x = p*q in A.

'Prime Object'

An object x∈ B – $\{1_A\}$ is called a 'Prime Object' in B if x is not a composite object in B.

It may be noted that any composite or prime object in B must be a member of B. By construction here, there is no reason to check whether the element 0_A and the self-inverse elements (other than 1_A) of the region A are 'prime' or 'composite' or 'neither prime nor composite' in any bachelor set in the region A, as they can not be members of any bachelor set in A. However, 1_A is the only element in any bachelor B which is neither a prime object nor a composite object. For every other object x (i.e. if x ≠1_A) in B, x is by default either a prime object or a composite object. Thus the following proposition is straightforward.

Proposition 2.1.4

There can not be any object x in the bachelor B in the region A which is both prime and composite.

If may be noted here that an object x may be prime in a bachelor B of a region A, but may not be so in another bachelor C of the same region A, even if x∈ B, C both. Thus, for a given region, the property of prime, composite and 'neither prime nor composite' is dependent upon the concerned bachelor set, and they must be members of the concerned bachelor set. For a given bachelor set, checking an object of a region whether prime or composite or 'neither prime nor composite' with respect to this bachelor set is an invalid issue if the object itself be not a member of the bachelor set.

A Partition of a Bachelor Set

For a bachelor set V in a region A, an important partition of the set V can be made into three subsets **:** the set of Prime objects in V, the set of Composite objects in V, and the set of neither Prime nor Composite objects in V, as shown in Figure 1. This is a partition of the bachelor set V because there can not be any object in the set V which is both a prime object and a composite object simultaneously in V.

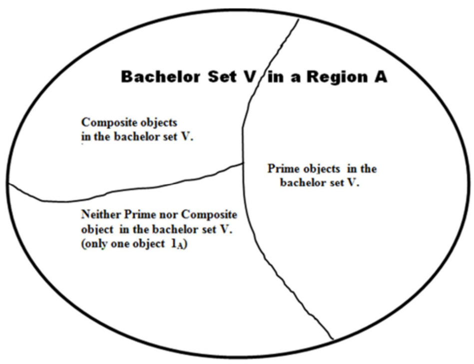


Fig. 1. *Prime, Composite and 'neither prime nor composite' objects in a bachelor set V in the region A.*

The following proposition is now straightforward.

Proposition 2.1.5

If x is a prime (composite) object in a bachelor B of a region R, then x^{-1} is a prime (composite) object in the conjugate bachelor $\tilde{B}$ and conversely.

We present below examples of the notion of prime objects and composite objects in a bachelor set in a region.

Example 2.1.3

Consider the region RR. Consider the bachelor set N of the region RR where N = {1, 2, 3, 4, 5, 6, 7, 8, …} = the set of natural numbers. Clearly, the members 4, 6, 8, 9, 10, 12, 14,…. are composite objects of the bachelor N here in the region RR; and the members 2, 3, 5, 7, 11, 13,…. are prime objects of the bachelor N in RR. ***Actually these are popularly known as 'composite numbers' and 'prime numbers' respectively in the existing literature of the classical 'Theory of Numbers'.*** There can not be any object in the bachelor N which is both prime and composite.

And 1 is the only object in the bachelor N which is neither a prime object nor a composite object (see Figure 2). There is no object in the bachelor N which is both prime and composite object. In fact this is a very much known result in the existing classical 'Theory of Numbers' that the integer 1 is neither a prime number nor a composite number.

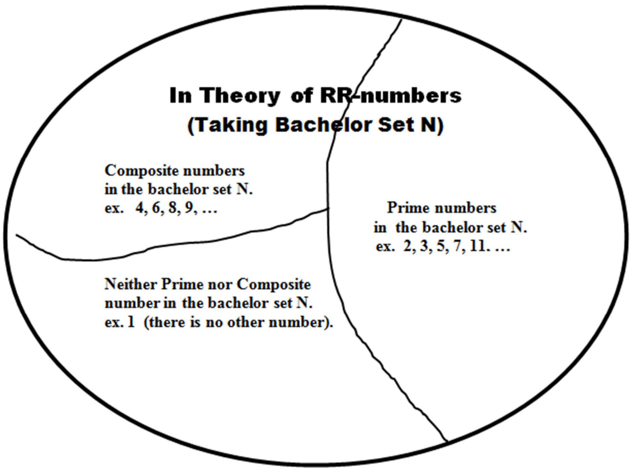


Fig. 2. *Prime, Composite and 'neither prime nor composite' numbers in the bachelor set N (of natural numbers) in the region RR.*

Another example of prime and composite objects is given below.

Example 2.1.4

Consider the region RR. Consider the bachelor set M of the region RR where M = {1, 1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, …} = { m : m = 1/n, n ∈ N, where N is the set of natural numbers}. Clearly, the members 1/4, 1/6, 1/8, 1/9, 1/10, 1/12,…. are composite objects of the bachelor M here in the region RR; and the members 1/2, 1/3, 1/5, 1/7, 1/11, 1/13,…. are prime objects of M in RR (see Figure 3). And 1 is the only object in the bachelor M which is neither a prime object nor a composite object. There is no object in the bachelor M which is both prime and composite.

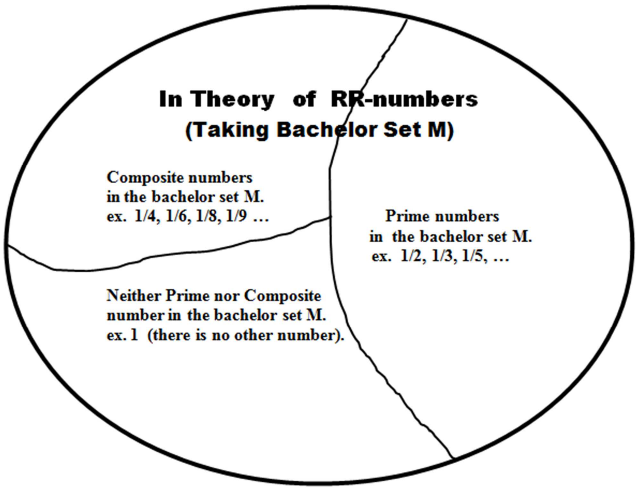


Fig. 3. *Prime, Composite and 'neither prime nor composite' numbers in the bachelor set M in the region RR.*

Example 2.1.5

Consider the bachelor L = {1, 78.261, 9287, 83.5} of the region RR. Clearly, the members 78.261, 9287, 83.5 are prime objects in the bachelor L; there does not exist any composite object in L. And 1 is the only object in the bachelor L which is neither prime object nor composite object. There can not be any object in the bachelor L which is both prime and composite.

The above Examples show that the classical prime numbers (in the existing classical 'Theory of Numbers') are particular case of prime objects in the region RR with respect to its bachelor set N. It may be noted that the notion of prime objects and composite objects are defined over any region, need not necessarily be in a complete region.

2.2. "Imaginary Objects" and "Complex Objects" of a Region

In this section we introduce the topic of 'imaginary objects' of a region. However, we will also see here that a region A may or may not have imaginary object. A region even may have more than one imaginary objects too. Imaginary objects of a region A are not members of A and so they are called 'imaginary' with respect to the concerned region A only (i.e. it is a local characteristics property with respect to the region concerned). An imaginary object of a region A could be core member (not imaginary object) of many other regions. Every region has its own set of imaginary objects (if exist).

2.2.1. 'Existence' of Imaginary Objects and Complex Objects of a Region

Consider a region A = (A, ⊕ ,*, •). For the region A, any member of the set A is called a "real object" of the region A. If something is not a member of the region A, we can not call it a real object of the region A.

I. Existence of imaginary object of a region

Let $E_1(x)$ and E_2 (x) be two single variable expressions valid in the region A. (It may be recalled that an expression is regarded to be a valid expression in an algebraic system A if it can be computed in A with the valid operations of A). If the equality (not identity) $E_1(x) = E_2(x)$ is not satisfied by any element x of the region A, then we say that the region A has at least one "imaginary object".

Let us imagine that i is an imaginary object of the region A coming out of the equation $E_1(x) = E_2(x)$. Then we must have $E_1(i) = E_2(i) = z$ (say) where z is a member of A, but i is not a member of A. Let us designate this imaginary object i of A to be an atomic imaginary object. Then any expression $E(i, x_1, x_2, x_3, \ldots, x_n)$ with respect to the operations $\oplus$,*,• of the region A over its outer field F is called a "complex object" of the region A. There may exist nil or one or more number of atomic imaginary object in a region A, and corresponding to every imaginary object (if exists) there exists a set of complex objects of the region A.

It may be noted here that by definition we can only realize about the existence of an imaginary object of a region A, but we can not trace its identity immediately. Because, an imaginary object of a region A is not a member of A, and at the same time it is fact that, on this issue we officially know nothing beyond the boundary of the set A at this stage. It is an open problem to us for further study and research.

Example 2.2.1

Consider the region RR. If we take $E_1(x) = x^2 +1$ and $E_2(x) = 2x - 1$, then we understand the existence of at least one imaginary object of the region RR.

If we take $E_1(x) = x^2 +1$ and $E_2(x) = 0$, then this too shows that the RR region does have at least one imaginary object. But, by the above examples, we are not sure here whether there exist only finite number or infinite number of imaginary objects of RR.

Example 2.2.2

In the simple trivial region (Z_2, $\oplus$,.,.) where $Z_2 = \{0, 1\}$, $\oplus$ is the "addition modulus 2" operator and '.' is the 'multiplication modulus 2' of real numbers, we see that if we take $E_1(x) = 2x+1$ and $E_2(x) = 0$, then we observe that there exist at least one imaginary object for this region Z_2.

However, if we take $E_1(x) = x^2 +1$ and $E_2(x) = 0$ then it does not help us to know the existence of any imaginary object of Z_2.

It is justified in Subsection-4.9 in [11] that there exist infinite number of distinct 1-D complete regions mathematically.

Proposition 2.2.1

Every complete region with characteristic zero has at least one imaginary object.

Proof. (This proposition is established subsequently in Proposition 2.4.2 here).

II. $\ni_A$-Complex Objects

Consider any complete region A = (A, $\oplus$,*, •) whose characteristic is zero. It is justified above that every such region has at least one imaginary object. Consider any imaginary object of A, which is $\ni_A$ (say). Then $\forall\ x_A,\ y_A \in$ A, the object ($x_A \oplus \ni_A y_A$) is called an " $\ni_A$-complex object" corresponding to the region A. In that case the object x_A is called the 'real part' and the object y_A is called the 'imaginary part' of the $\ni_A$-complex object. Obviously both real part and imaginary part of an $\ni_A$-complex object are real objects of the region A.

III. $\ni$-Complex Objects : a particular case of $\ni_A$-Complex Objects

Consider the infinite region A = (A, $\oplus$,*, •) whose characteristic is zero. In Proposition 2.4.2 (established subsequently), it is shown that the equation $x_A^2 + 1_A = 0_A$ is not satisfied by any object of A. Suppose that the corresponding particular imaginary object $\ni_A$ is denoted by the notation $\ni$.

Thus we have the result $\ni^2 + 1_A = 0_A$, i.e. $\ni^2 = \sim 1_A$.

Then $\forall\ x_A,\ y_A \in$ A, the object ($x_A \oplus \ni y_A$) is called an $\ni$-complex object corresponding to the region A.

The set $C_A = \{ z_A = (x_A \oplus \ni y_A) : x_A,\ y_A \in A\}$ is called the set of all $\ni$-complex objects corresponding to the region A.

IV. Algebra of $\ni$-Complex Objects

Consider the set C_A of all $\ni$-complex objects corresponding to the infinite region A. Denote the $\ni$-complex object ($0_A \oplus \ni 0_A$) by the notation $\ni 0$ and the $\ni$-complex object ($1_A \oplus \ni 0_A$) by the notation $\ni 1$. If $z_A = (x_A \oplus \ni y_A)$ be an $\ni$-complex object, then we define its conjugate $\ni$-complex object given by $\tilde{z_A} = (x_A \sim \ni y_A)$.

Define the following operations over the set C_A :

(1) Addition & Subtraction

If $z_{A1} = (x_{A1} \oplus \ni y_{A1})$ and $z_{A2} = (x_{A2} \oplus \ni y_{A2})$ be two $\ni$-complex objects, then define addition of them using the identical notation $\oplus$ as below

$z_{A1} \oplus z_{A2} = (x_{A1} \oplus x_{A2}) \oplus \ni (y_{A1} \oplus y_{A2})$, which clearly belongs to C_A;

and define subtraction as below

$z_{A1} \sim z_{A2} = (x_{A1} \sim x_{A2}) \oplus \ni (y_{A1} \sim y_{A2})$, which clearly belongs to C_A.

(2) Multiplication

If $z_{A1} = (x_{A1} \oplus \ni y_{A1})$ and $z_{A2} = (x_{A2} \oplus \ni y_{A2})$ be two $\ni$-complex objects, then define multiplication of them using the identical notation $*$ as below

$z_{A1} * z_{A2} = (x_{A1} * x_{A2} \sim y_{A1} * y_{A2}) \oplus \ni (x_{A1} * y_{A2} \oplus y_{A1} * x_{A2})$ which clearly belongs to C_A.

(3) Scalar Multiplication

For $k \in R$ and for $z_A = (x_A \oplus \ni y_A) \in C_A$, define the scalar multiplication as below:

$k \bullet z_A = (k \bullet x_A \oplus \ni k \bullet y_A)$, which clearly belongs to C_A.

V. Im-numbers and Imaginary Numbers

The 'imaginary objects' of the particular region RR are to be called by 'imaginary numbers' in our Region Mathematics. But now there arises a conflict (of title) because of the fact that the existing 'Theory of Numbers' has also a notion of 'imaginary numbers'. To avoid confusion between the existing concept of 'imaginary numbers' and our notion of 'imaginary numbers' which is introduced here for the region RR, we will henceforth call our notion of 'imaginary numbers' by the abbreviated term 'im-numbers'.

It is obvious that all the imaginary numbers are im-numbers, but at this moment we can not answer whether the converse is true or not.

We call the im-numbers for the set of real numbers R by the term R-im or rim (in short). If there exist im-numbers for the set of complex numbers C then we will call each of them by the term C-im or cim (in short). The existing 'Theory of Numbers' says that i is a rim.

2.2.2. "Square Root" of an Object in a Region

For a given object z of a region A, if $\exists$ $x \in A$ such that $x^2 = z$ then we say that x is a real square root object (or, simply may be called 'square root') of the object z, denoted by $\sqrt{z} = x$.

An object of a region A may have nil or more number of real square roots. Clearly 0_A and 1_A are the only objects for which the object itself is the square root of it respectively. However 1_A may have more than one square roots.

Example 2.2.3

Consider the region RR. Clearly the object 9 of RR has a square root and the object -9 does not have any square root. Hence -9 has at least one imaginary square root. It implies that the region RR does have at least one imaginary object.

2.2.3. "nth Root" of an Object in a Region

For a given object z of a region A, if $\exists$ $x \in A$ such that $x^n = z$ then we say that x is a real nth root object (or, simply may be called 'nth root') of the object z denoted by $\sqrt[n]{z}$, where n is a positive integer. An object may have nil or more number of real nth roots. In case, for a given z the equation $x^n = z$ is not satisfied by any $x \in A$, then we say that z has at least one 'imaginary nth root'; and at the same time we understand the existence of at least one 'imaginary object' of the region A.

2.2.4. Classical Set of "Complex Numbers": A particular Instance

For an arbitrary region A, knowing about the possible 'existence' of some imaginary objects of it is not a straightforward task. Consequently, knowing the 'identities' of the imaginary objects of it (if exist) is also not a straightforward task, unlike knowing the imaginary objects of the region RR which is a particular case. Nevertheless, according to our Theory of Objects there is no guarantee at this stage that: "the set of all imaginary objects of the region RR is exactly equal to the set of complex numbers". It is an open problem now for us. However it is now guaranteed that the classical set C of complex numbers is a subset of the set of all imaginary objects of the region RR.

2.2.5. Logarithm of Objects

Consider a region A. For two objects x and y of the region A, the logarithm of an object x to the base y is denoted by the notation $\log_y(x)$ is the unique real number b such that $y^b = x$. We will discuss the issue for $x = 0_A$ or for $y = 0_A$ later on, but in an analogous way of classical logarithm results. We will also establish the classical algebraic results of logarithm in Region Mathematics in later subsections. We will see that if A and B are two distinct complete regions, then the real numbers like $\log_{2_A} 4_A$ (i.e. logarithm of the object 4_A to the base 2_A) and $\log_{2_B} 4_B$ are not equal. The objects like 4_A, 4_B etc. are introduced in subsection 2.4 below.

2.3. Discovery of "Compound Numbers": Another New Direction Unearthed in the Classical 'Theory of Numbers'

Take the function $f(x) = x^2 + 1$. There is no x in the region RR (set R) which satisfies the equation $f(x) = 0$. It indicates that there is at least one rim in R. It is in fact well known to everybody that R has one rim which is i. At this moment we will not debate on the issue "How many distinct atomic rims R does have of kind i", unless we do further work on it in the context of region mathematics. As in the existing literatures on the classical Theory of Numbers, there is one and only one atomic rim which is i, of course along with infinite number of other rims of kind (a+ib).

Now let us consider the following analysis very carefully:

Consider the region C. Consider the function $f: C \rightarrow C$ given by

$$f(z) = (|z|^2 + 2) + 3i.$$

Consider another function $g : C \rightarrow C$ given by

$$g(z) = 1 + 3i.$$

It may be observed that there is no object z of C which satisfies the equation $f(z) = g(z)$. Consequently, according to our earlier discussion made in subsection-2.2.1, it indicates that there is at least one imaginary object (cim) in C. Say e is one atomic cim in C generated from the above equation $f(z) = g(z)$. It means that e is an imaginary object of C for which the equality $f(e) = g(e) = z_0$ holds good, where $z_0 \in C$.

It is to be noted that i is an im-member of the region R, not of the region C; and similarly e is an imaginary member of the region C, not of the region R. Thus for the real objects z_1 and z_2 of C, if e is one cim of C then the object $d = (z_1 + e\, z_2)$ is not a member in C i.e. is not a real object of C (this situation is analogous to the case where for x_1 and x_2 of R, the object $d = (x_1 + i\, x_2)$ is not a member in R). Such an object $d = (z_1 + e\, z_2)$ is an complex object of the region C and is called by a **compound number** in C here. The complex number z_1 is a real object of C and is called the 'complex part' of the compound number d; and the complex number z_2 is also a real object of C and is called the 'imaginary part' of the compound number d. Corresponding to every atomic cim, there exist infinite number of compound numbers.

In general, suppose that R_1, R_2, R_3, ….., R_n are n number of regions. A region may or may not have imaginary object. Even if a region R_i has an imaginary object, we need to explore how many more imaginary object does R_i have. If e_i is an imaginary object of the region R_i and if a, b are real objects of R_i then $(a + b\, e_i)$ is a complex object of the region R_i. However, for the particular region C, its complex objects are called by compound numbers.

No confusion about the existence of cim

If x, y are in R then the equation $x^2 + y^2 + 1 = 0$ is not satisfied by any x, y of R and thus there may exist one or more solutions of this equation in the form of $x = x_1 + i\, x_2$ which are imaginary objects of the region R (which we know as complex numbers). The equation f(z) = g(z) where $f(z) = (|z|^2 + 2) + 3i$ and $g(z) = 1 + 3i$ can not be solved for z in C. This situation leads to the existence of at least one cim. Consequently, it is to be very carefully noted that although searching for x and y from R for satisfying the equation $x^2 + y^2 + 1 = 0$ and searching for z satisfying the equation f(z) = g(z) where $f(z) = (|z|^2 + 2) + 3i$ and $g(z) = 1 + 3i$ are basically same type of problems, but these two searching are to be executed on two different platforms (two different regions). In the first case we do search for real numbers x and y from the jurisdiction R only, whereas in the second case we do search for a complex number z from the jurisdiction C only. We must be careful about our boundary of the concerned region while searching for solutions of valid equations in that region. Thus, there is no confusion in the existence of at least one atomic cim of C, but its precise identification and details characterization are to be done.

History says that after the discovery of the rim i, a new number system took shape which is the set C of complex numbers. It is to be philosophically viewed that the existing notion of ‘complex numbers’ is with respect to its base ‘real numbers’. In this sense ‘5i’ is an imaginary number to the set R, not to the set C. To the set C the number ‘5i’ is a core family-member. It is to be clearly understood that the issue of ‘imaginary’ or ‘complex’ is an relative issue, but local to the concerned region. One object may be a core family (not an imaginary object) to a region A, but it could be an imaginary object to another region B (not a core family member)!.

Thus our history says that the set R of real numbers is conceptualized first, and later by the discovery of i the mathematicians discovered the birth of the classical set C of complex numbers. In an analogous way we claim that picking-up the region C and by the discovery of the cim e (and other atomic cims, if exist of C) has led to the discovery of a new set of numbers. Let us call this new set by the set of “Compound Numbers” denoted by E. Our immediate need is to discover the fundamental operations on E (like additions, multiplications, etc.) and then to study E as a possible algebra, and more. It is obvious that E forms a group with respect to the binary operator ‘+’ defined as below :
for the compound numbers $d_1 = z_{11} + e\, z_{12}$ and $d_2= z_{21} + e\, z_{22}$ of E, define $(d_1+ d_2)$ by $d_1+ d_2 = (z_{11}+ z_{21}) + e\,(z_{12} + z_{22})$, which is obviously a compound number in E.

In the “Theory of Objects”, the set E of Compound Numbers introduced here is presently just at its infant stage, but undoubtedly it is a new set of numbers discovered here. With a rigorous amount of research work on the set E of numbers, it will surely take its own shape in future to update the existing classical “Theory of Numbers”. Without giving further justifications, we claim that there are many more sets of numbers (besides the set E of numbers) yet to be discovered. The next subsection will now show another new direction about the existence of many new sets of numbers.

Subsequently in due time, by the discovery of many new sets of numbers more, we need to revisit many of the existing famous results, viz:

(i) R, C, H, O are the only normed division algebras.

(ii) the only associative real division algebras are real numbers, complex numbers, and quaternions.

(iii) The Cayley algebra is the only non-associative division algebra.

(iv) The algebras of real numbers, complex numbers, quaternions, and Cayley numbers are the only ones where multiplication by unit "vectors" is distance- preserving.

2.4. “Theory of Numbers”: Every Complete Region Has Its Own

In the previous Subsection 2.1 and 2.2, while introducing the notion of prime and composite objects and then the notion of imaginary and compound objects, we have consider a general region which need not be a complete region (see Figure 4). There are regions which are complete and there are regions which are not. In this section we introduce a new type of “Theory of Numbers” corresponding to every complete region. It has been shown in subsection-4.9 in [11] that corresponding to a region A, there may exist infinite number of distinct complete regions (i.e. 1-D complete regions).

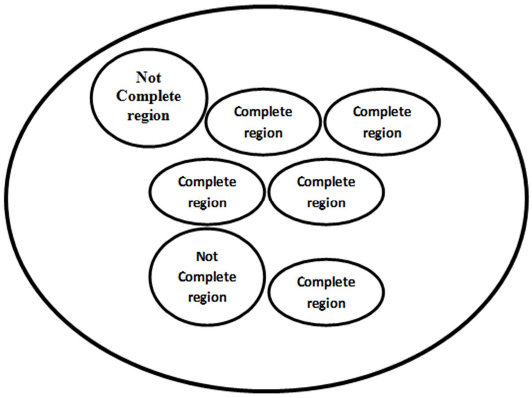


Fig. 4. *Collection of all regions (Complete and Not Complete).*

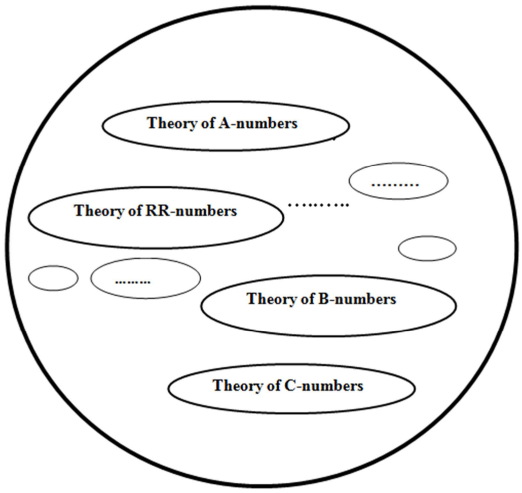


Fig. 5. *Collection of all complete regions, each has its own unique ‘Theory of Numbers’.*

2.4.1. "Theory of A-numbers" of a Complete Region A

In this subsection we develop a new theory called by "Theory of A-numbers" corresponding to a complete region A. Corresponding to every complete region A, we derive a corresponding unique new 'Theory of Numbers' called by **"Theory of A-numbers"**. If RR, A, B, C, … are the complete regions in region algebra, then the corresponding number theories are : "Theory of RR-numbers", "Theory of A-numbers", "Theory of B-numbers", "Theory of C-numbers",... etc. respectively, as shown in Figure 5. If a region K is not a complete region, the "Theory of K-numbers" does not exist for it; however the topics of prime and composite objects, imaginary and compound objects, can be well studied in any region K, be it a complete region or not.

I. Object Linear Continuum Line in a complete region A

The notion of 'Object Linear Continuum Line' in a complete region A is explained earlier in details in Section-4 in [11]. It is mentioned earlier that by a complete region, we mean 1-D complete region. A line can be drawn on which one point may be fixed to be the location for the object 0_A, with all positive objects of A having their respective locations to the right and all negative objects of A having their respective locations to the left of 0_A, as explained in subsection 4.1 and subsection 4.6 in our earlier work in [11]. Thus the 'positive direction' of the line can be called to be X_A-axis and the 'negative direction' of the line can be called to be X_A^1-axis. And the line which the objects of the complete region A is considered to lie upon is called the Object Linear Continuum Line for the complete region A (see Figure 6).

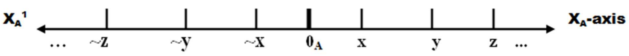


Fig. 6. *Object Linear Continuum Line of the complete region A, a general view.*

By distance between two objects x and y of the complete region A, we mean the corresponding metric distance ρ (x,y) of the normed complete metric space A. The distance of a positive object x_A from the origin is $\|x_A\| = \rho(x_A, 0_A) = x_a$, and the distance of a negative object ~ x_A from the origin is = $-x_a$. See a collection of consecutive equi-spaced points on the object line as shown in the Figure 7 below.

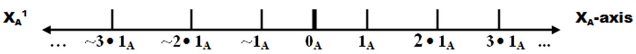


Fig. 7. *Object Linear Continuum Line of the complete region A with a collection of consecutive equi-spaced object points.*

The term 'equi-spaced' in the caption of Figure 7 is to be well understood in the sense of the corresponding metric (or norm) of the complete region A, i.e. for any real number r, ρ (r • 1_A, (r+1) • 1_A) = positive constant (independent of the real number r), in the complete region A. But this constant real value is different for different complete regions. For the complete region RR, this constant real value is equal to the real number 1.

Since A = (A, ⊕ ,*, •) is complete (normed complete metric space), there are no "points missing" from it (inside or at the boundary). Since A is a chain, every object of A has a unique address on this Object Linear Continuum Line $X_A{}^1X_A$; and conversely i.e. corresponding to every address (point) on this Object Linear Continuum Line $X_A{}^1X_A$ there is a unique object of the region A.

II 'Unit Length' & 'Inverse Unit Length' in a complete region A

Consider a complete region A = (A, ⊕ ,*, •). If x_A is a positive object on the object linear continuum line, then the distance of x_A from the point O (the location of the object 0_A on the object linear continuum line $X_A{}^1X_A$) is denoted by x_a which is a positive real number (we use the classical practiced convention to say that ~x_A is at a distance of $-x_a$ from the point O, although as per definition of metric a distance can not be a negative quantity.

For $x_A \in$ A, we have

$$\|x_A\| = \begin{cases} x_a & \text{if } x_A \text{ is a positive object} \\ -x_a & \text{if } x_A \text{ is a negative object} \end{cases}$$

because $\rho(0_A, x_A) = \rho(0_A, \sim x_A) = |x_a|$.

Corresponding to the unit element 1_A of the complete region A, the positive real number 1_a (where $1_a = \|1_A\| = \rho$ $(0_A, 1_A)$) is called the 'unit length' in the Theory of A-numbers. Surely 'unit length' is a real number and is constant in the 'Theory of A-numbers', but may be different for different complete regions. That is, 'unit length' of the 'Theory of A-numbers' is not necessarily equal to the 'unit length' of the 'Theory of B-numbers'. Thus, if A, B, C, …… are complete regions, then the respective unit lengths 1_a, 1_b, 1_c,...... are not equal in general. Clearly 0_a being the $\|0_A\|$ is equal to the real number 0, and it may also be noted here that in general $1_a \neq 1$ (where 0 is the 0_{RR} i.e. 0 is the classical real number 0 of R; and 1 is the 1_{RR} i.e 1 is the classical real number 1 of R).

Suppose that $1/1_a = ə_a$ (see the 'division Type-IV' in the four types of division in a region algebra explained in subsection 3.2.9 in [11]). The positive real number $ə_a$ is called the 'Inverse Unit Length' in the complete region A. Surely 'inverse unit length' is a real number and is constant in the 'Theory of A-numbers', but may be different for different complete regions. That is, 'inverse unit length' of the 'Theory of A-numbers' is not necessarily equal to the 'inverse unit length' of the 'Theory of B-numbers'. It is also obvious that neither the unit length nor the inverse unit length can be equal to 0 in the 'Theory of A-numbers' corresponding to any complete region A. Clearly $1_a.\ ə_a = 1$ and in general $1_a \neq ə_a$ in the complete region A. However, for the particular complete region RR, we have $1_{rr} = ə_{rr} = 1$ (= 1_{RR}) which is one of the most beautiful properties of the set R of real numbers, and also it is one of the most powerful properties of R. Out of several reasons, this property is one of the leading property due to which most of our real life problems and practices suit the set R of real numbers.

III 'Ontegers' in the complete region A

In this subsection we introduce the notion of 'Onteger' in a complete region A = (A, ⊕ ,*, •). The word 'onteger' is not a valid word in English dictionary. It is an abbreviated word for "Object Integer". The concept of 'ontegers' will be the basic element in developing the 'Theory of A-numbers'.

Consider an object x_A in the complete region A. Consider the real number $x_a/1_a$ i.e. $x_a.\ ə_a$ which let us denote by the symbol x. Thus x ia a real number given by $x = x_a/1_a = x_a.\ ə_a$.

Thus $x_A = x \bullet 1_A$, $\|x_A\| = x_a$ where $x_a = x.1_a \ \forall\ x_A \in A$ where x_A is a positive object; and $\sim x_A = -x \bullet 1_A$, $\|\sim x_A\| = x_a$ where $x_a = x.1_a\ \forall\ \sim x_A \in A$ but $\sim x_A$ is a negative object.

It may be noted here that in general $1_a \neq 1$, and in a similar way $x_a \neq x$. However it is obvious that for the particular instance of the complete region RR, we have $x_{rr} = x\ (= x_{RR})$.

Onteger

If m is any real integer, then the object m_A of the complete region A is called an 'object integer' or 'onteger' in the 'Theory of A-numbers'.

Thus the ontegers in the 'Theory of A-numbers' are 0_A, ⊕ 1_A, $\sim 1_A$, ⊕ 2_A, $\sim 2_A$, ⊕ 3_A, $\sim 3_A$, ….. etc. The ontegers ⊕ 1_A, ⊕ 2_A, ⊕ 3_A, ⊕ 4_A, …….. etc. are 'positive ontegers' and the ontegers $\sim 1_A$, $\sim 2_A$, $\sim 3_A$, $\sim 4_A$, …….. etc. are 'negative ontegers' in the 'Theory of A-numbers'. The onteger 0_A is neither a positive onteger nor a negative onteger. Obviously, the set of all ontegers of the complete region A is a countable set. However, it may be true that norm of some of the ontegers of the complete region A are integers in R. In a complete region A, the ontegers 0_A, ⊕ 2_A, $\sim 2_A$, ⊕ 4_A, $\sim 4_A$, …..are even ontegers and the ontegers ⊕ 1_A, $\sim 1_A$, ⊕ 3_A, $\sim 3_A$ ⊕ 5_A, $\sim 5_A$ …….. are odd ontegers.

It is to be carefully noted that corresponding to any onteger ⊕ m_A of the complete region A, the distance m_a of it from the point 0_A on the object line is a real number but need not necessarily is a real integer of classical notion; and similarly corresponding to any onteger $\sim m_A$, the distance $-m_a$ is a real number but not necessarily is a real integer. The classical notion of integers in the classical 'Theory of Numbers' is just a special case of ontegers which is corresponding to the "Theory of RR-numbers'. Corresponding to every complete region A, there is a unique "Theory of A-numbers". Consider the complete regions RR, A, B, C, D, … etc. and the corresponding "Theory of RR-numbers", "Theory of A-numbers", "Theory of B-numbers", "Theory of C-numbers", "Theory of D-numbers"… respectively. If we imagine one common Object Linear Continuum Line for all these different complete regions RR, A, B, C, D, ….. etc. with the respective zero elements 0, 0_A, 0_B, 0_C, 0_D, … being situated at exactly the same point on this common Object Linear Continuum Line, then it is obvious that the respective unit elements $1(1_{RR})$, 1_A, 1_B, 1_C, 1_D, … etc. will be situated in general at different points on the common line because of the fact that the 'unit length' is region-dependent (see Figure 8)

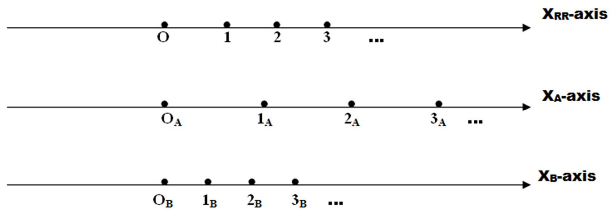


Fig. 8. *Ontegers for "Theory of RR-numbers", "Theory of A-numbers" and "Theory of B-numbers" (a comparative view).*

Thus, for any given real number x, in general the points x, x_A, x_B, x_C, x_D, … etc. will be situated at different locations if placed on the common Object Linear Continuum Line, assuming that the respective zero elements 0, 0_A, 0_B, 0_C, 0_D, … are situated at exactly the same coincident point on this common Object Linear Continuum Line.

However, for a given complete region A the distance between two consecutive ontegers on the object linear continuum line will be always a constant real number; but this real constant value will be different for different complete regions.

Thus, we have for the complete region A,

$$\ldots = \rho(\sim 3_A, \sim 2_A) = \rho(\sim 2_A, \sim 1_A) = \rho(\sim 1_A, 0_A) = 1_a = \rho(0_A, 1_A) = \rho(1_A, 2_A) = \rho(2_A, 3_A) = \rho(3_A, 4_A) = \ldots,$$

and similarly for the complete region B,

$$\ldots = \rho(\sim 3_B, \sim 2_B) = \rho(\sim 2_B, \sim 1_B) = \rho(\sim 1_B, 0_B) = 1_b = \rho(0_B, 1_B) = \rho(1_B, 2_B) = \rho(2_B, 3_B) = \rho(3_B, 4_B) = \ldots.$$

But, in general, $\rho(0_A, 1_A) \neq \rho(0_B, 1_B)$.

(here it is needless to mention that the metric ρ for the complete region A and the metric ρ for the complete region B are two different metrics, in general. There should not be any confusion in it).

For any real number r, $\rho(r \bullet 1_A, (r+1) \bullet 1_A)$ = positive constant 1_a (independent of the real number r), in the complete region A. But this constant real value is different for different complete regions because of the fact that the real numbers 1_a, 1_b, 1_c, ….. are not necessarily equal. For the complete region RR, this constant real value is equal to the number 1. For any real numbers r and k, $\rho(r \bullet 1_A, (r+k) \bullet 1_A) = |k|.1_a$.

On the RR region line i.e. on the real number line, distance of the object ⊕ 1_{RR} or $\sim 1_{RR}$ from the object 0_{RR} (i.e. distance of the real number +1 or -1 from the number 0) is of 'unit length' and is popularly called by us by the word 'unit' or 'one'. It may be recalled that for every $x_A \in A$, x_a is in R.

Consider a common object linear continuum line for both the "Theory of RR-Numbers" and the "Theory of A-Numbers", with 0_{RR} and 0_A situated at a coincident point on the line. Then it may happen that the point on the line representing the integer 1 (onteger 1_{RR}) is not the point representing an onteger of the "Theory of A-Numbers". Conversely, it may happen that the point on the line representing the onteger 1_A of the "Theory of A-Numbers" is not the point representing an integer of the "Theory of RR-

Numbers". The main source of such differences lies in the difference of size of 'unit length' of different complete regions in the "Theory of Objects".

Thus 'Theory of A-numbers' is different for different complete region A in the "Theory of Objects", whereas the complete vast amount of the classical 'Theory of Numbers' being available in the existing literature and being practiced by us traditionally so far in our everyday life is just a fractional content of the 'Theory of RR-numbers' in the context of the "Theory of Objects".

The following proposition is now straightforward and a quite important result in the 'Theory of A-numbers'.

Proposition 2.4.1

Corresponding to a real number x (-x), there is a unique object $\oplus x_A$ ($\sim x_A$) in the complete region A and hence a unique corresponding real number x_a ($-x_a$) which in general, not equal to the real number x (-x).

IV. 'R_A value' of a real number x

Let A be a complete region. Corresponding to the complete region A, consider the 1-to-1 mapping $R_A : R \rightarrow R$ defined by

$R_A(x) = x.1_a = x_a \; \forall \; x \in R$. Then the real number x_a is called the 'R_A value' of the real number x denoted by $R_A(x) = x_a$ corresponding to the complete region A. Clearly, in that case $R_A(-x) = -x_a$. Also $R_A(0) = 0_a$, and $R_A(1) = 1_a$.

For $x_A \in A$, we have

$$\|x_A\| = |R_A(x)| = \begin{cases} x_a & \textit{if } x_A \textit{ is a positive object} \\ -x_a & \textit{if } x_A \textit{ is a negative object} \end{cases}$$

because $\rho(0_A, x_A) = \rho(0_A, \sim x_A) = |x_a|$.

It is obvious that R_{RR}: $R \rightarrow R$ is an identity mapping for the particular case of the complete region RR.

V. 'Set of R values' and 'Set of R objects' corresponding to a real number x in the "Theory of Objects".

If RR, A, B, C, D, … are the complete regions in the complete region universe Σ, then for any given real number x the set $\Sigma_x = \{x_{rr} (= x), x_a, x_b, x_c, x_d, \ldots\ldots\}$ is called the 'Set of R values' of the real number x in the region universe Σ. And the set $\Sigma_X = \{x_{RR} (= x), x_A, x_B, x_C, x_D, \ldots\ldots\}$ is called the 'Set of R objects' of the real number x in the region universe Σ.

Although we call Σ_x a set, it is a mistake because Σ_x could be a multiset (bag) too. However, if there is no confusion we will continue here calling it a set. For details about the concept of distance (metric space) introduced in multisets, one could see [10]. Thus, the collection of R values of the real number 1 is the set (multiset) of all unit length values forming Σ_1, and the collection of R values of the real number 0 is the set (multiset) Σ_0.

VI. Natural A-ontegers' and 'Natural A-real numbers' of a complete region A

In the Theory of A-numbers, the positive ontegers $\oplus 1_A$, $\oplus 2_A$, $\oplus 3_A$, $\oplus 4_A$, … are called the Natural A-ontegers and the corresponding positive real numbers 1_a, 2_a, 3_a, 4_a, … are called the 'Natural A-real numbers'.

For instance, in the 'Theory of RR-numbers', the Natural RR-ontegers are 1_{RR}, 2_{RR}, 3_{RR}, 4_{RR}, … and the Natural RR-real numbers are 1_{rr}, 2_{rr}, 3_{rr}, 4_{rr}, …. Here, as a particular case, the Natural RR-ontegers and Natural RR-real numbers are same numbers i.e. the classical natural numbers 1, 2, 3, 4, …..

Let us consider three complete regions RR, A and B. The natural RR-real numbers (i.e. the classical natural numbers), natural A-real numbers and natural B-real numbers, all being real numbers only, can be plotted on the real line. It can be observed that the consecutive natural RR-real numbers are equi-spaced on the real line, and the same is true for consecutive natural A-real numbers on the real line i.e. the consecutive natural A-real numbers are equi-spaced on the real line. Similarly, the consecutive natural B-real numbers are equi-spaced on the real line. But the unit lengths are different for different object linear continuum lines corresponding to different complete regions.

Consider the positive real number $x \in R$, and any three complete regions say RR, A and B. The corresponding three objects of the set Σ_X are given by: x_{RR} (i.e. x itself) on the X_{RR}-axis, x_A on the X_A-axis, and x_B on the X_B-axis are shown in Figure 9, assuming that the respective zero elements 0_{RR} (i.e. 0), 0_A and 0_B are situated at exactly the same coincident point on a common object line. Corresponding to a pre-fixed real number x, the three elements x_{RR}, x_A and x_B of Σ_X in the three theories : "Theory of RR-numbers", "Theory of A-numbers" and "Theory of B-numbers", are shown in Figure 9 for a comparative view.

On their respective axis of linear continuum, the object x is at a distance x from O_{RR} (i.e. O), the object x_A is at a distance x_a from O_A and the object x_B is at a distance x_b from O_B. Here $x_{rr} = x.1_{rr}$ (where $x_{rr} = x$ and $1_{rr} = 1$), $x_a = x.1_a$, and $x_b = x.1_b$. The three real numbers

(i) 1_a (the metric distance of the unit object 1_A from the centre point O_A),
(ii) 1_b (the metric distance of the unit object 1_B from the centre point O_B) and
(iii) 1_{rr} or 1 (the metric distance of the unit object 1_{RR} from the centre point O_{RR})

are not equal in general, a hypothetical case about which can be realized from the Figure 8, where $1_b < 1_{rr} (=1) < 1_a$.

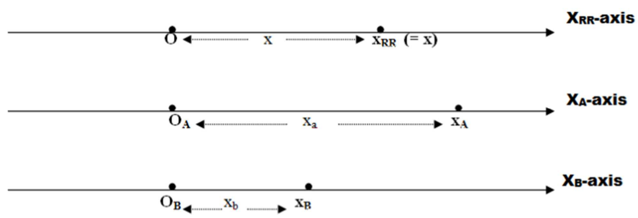


Fig. 9. *Three elements of Σ_X corresponding to common real number x in the three theories: "Theory of RR-numbers", "Theory of A-numbers" and "Theory of B-numbers" (a comparative view).*

It is justified in Subsection-4.9 in [11] that we can define infinite number of distinct 1-D complete regions mathematically. The proof of the earlier Proposition 2.2.1 is now presented in the following proposition.

Proposition 2.4.2

Every complete region with characteristic zero has at least one imaginary object.

Proof. Consider any complete region A = (A, $\oplus$,*, $\bullet$) whose characteristic is zero. In our literature, by complete region we mean 1-D region calculus. We take help of an example here. Consider the equation $x_A^2 + 1_A = 0_A$ in the region A.

We see that the equation $x_A^2 + 1_A = 0_A$ is not satisfied by any object of A, where both LHS and RHS of this equation are valid expressions in A. Let us prove it by contradiction.

i.e. if possible, suppose that for an object x_A of A we have

$$x_A^2 + 1_A = 0_A$$

$$\text{Or, } (x \bullet 1_A)^2 + 1_A = 0_A$$

$$\text{Or, } x^2 \bullet 1_A^2 + 1_A = 0_A$$

$$\text{Or, } (x^2 + 1) \bullet 1_A = 0_A$$

By Proposition 3.5 of [11], either $(x^2 + 1) = 0$ or $1_A = 0_A$ which is a contradiction. Therefore, there is no real object x_A of the region A which can satisfy the equation

$$x_A^2 + 1_A = 0_A$$

Consequently, it produces one imaginary object of the complete region A which is ∍ (say). Hence the result.

(Note: It may be seen that the equation $x_A^2 + 1_A = 0_A$ produces different imaginary objects for different complete regions A. It may also be noted that C does not form an 1-D region calculus (i.e. 1-D complete region), but it does not mean that C will not have any imaginary object.)

Proposition 2.4.3

If A is a complete region, then $\exists$ infinite set of trio x, y, z $\in$ A such that the relation $x^n \oplus y^n = z^n$ is satisfied for n = 2.

Proof. Take the case for x = 3_A, y = 4_A and z = 5_A.

$$3_A^2 \oplus 4_A^2 = (3 \bullet 1_A)^2 \oplus (4 \bullet 1_A)^2$$
$$= (9 \bullet 1_A^2 + 16 \bullet 1_A^2)$$
$$= 25 \bullet 1_A^2$$
$$= (5 \bullet 1_A)^2$$
$$= 5_A^2 \text{ Hence proved.}$$

2.4.2. "Theory of C-numbers" Can not Be Developed in C

It can be observed that 'Unique Factorization Theorem' holds good in the Theory of objects in any complete region A. The existing rich literature on Algebra or Number Theory does not say what is the minimum algebraic system on which such type of famous theorems of Number Theory hold good. In this work the same is identified to be a complete region. Otherwise the question does not arise to explore the above results of Number Theory and their validity. Many of the classical famous results of Number Theory are established during last three centuries but the platform on which they stand valid was not identified (although it is a very important information to the number theorists). These results can not be established in a division algebra or in any standard algebraic system alone, and even not in a region in general. It must be a complete region! This information is an important breakthrough for the Theory of Numbers. The existing Theory of Numbers (in particular, real numbers) are based upon the set R of real numbers which fortunately forms a complete region! And consequently the number theorists of the existing Theory of Numbers (real numbers) have not faced any issue or contradiction while fluently exercising all the properties of R at full freedom. The region C (of complex numbers) does not satisfy the required conditions to become a 1-D calculus space with respect to its popular norm |z| = $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. No 1-D region calculus can be developed in C, and hence C is not a 1-D complete region. Consequently, a number theory of type 'Theory of C-numbers' can not be developed in C, although the concept of prime and composite objects, imaginary and complex objects, etc. can be well studied in C. However, in our future work we need to explore whether C forms a multi-dimensional complete region; and if so then we may explore the possibility of discovering a number theory of type 'Theory of C-numbers' with multi-dimensional approach as mentioned in the section-4.8 in [11].

With the notion of "Theory of Objects" introduced here, it is sure that in due time the 'Number Theorists' can be re-designated with a new title 'Object Theorists' as the areas of cultivation will not be limited to just numbers but to the objects.

3. "Region Geometry": Every Complete Region Has Its Own

In Section-4.9 it is observed that we can define infinite number of distinct 1-D complete regions mathematically in Region Mathematics. It is also mentioned that by a complete region, we shall always mean 1-D complete region. In the previous section the notion of 'Theory of A-numbers' of a complete region A is developed. We are now in a position to initiate a corresponding 'Geometry' on the complete region A in the "Theory of Objects". The "Theory of Objects" thus induces a new giant area which is to be called here by 'Region Geometry' in a complete region. Corresponding to every complete region, there is a 'Region Geometry' of its own. But the 'Region Geometry' for different complete regions are different. We will see that our rich classical geometry of the existing notion is just a particular case of 'Region Geometry'. We begin the subject by introducing first of all a 2-D region geometry developed over an 1-D complete region.

In the Theory of Objects, for developing a new geometry called by "Region Geometry", be it in a two dimensional region coordinate system, or in an n-dimensional region coordinate system, at least one 1-D complete region A = (A, $\oplus$,*, $\bullet$) is required. The work in this section in fact is sequel to the previous Sections 4 of [11] and Section 2 of the present article. Consider the object linear continuum line and the X_A-axis corresponding to the complete region A. Consider a point x_A (a positive object) on the X_A-axis. Then for an infinitesimal small positive object Δx_A, the point $(x_A \oplus \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the positive direction of X_A-axis and the point $(x_A \sim \Delta x_A)$ will be at a distance $\|\Delta x_A\|$ from the point x_A along the negative direction $X_A{}^1$-axis; and in fact all the objects of the complete region A are well ordered in this sense, as explained in details in Section-4 in [11]. In this section we incorporate "Y_A-axis" (it is same as X_A-axis but placed at right angle to the X_A-axis passing through the point 0_A) and thus construct a region coordinate plane in the "Theory of Objects" in the style of Cartesian coordinate system.

3.1. The Coordinate Plane of Complete Region $A = (A, \oplus, *, \bullet)$

We introduce first of all 2-D region geometry in a 1-D complete region A = $(A, \oplus, *, \bullet)$. It is a system of geometry where the position of points on the plane is described using an ordered pair of objects, analogous to the case of Cartesian coordinate plane. We call this plane by Region Coordinate Plane. A plane is a flat surface that goes on forever in both directions. If we were to place a point on the plane, region coordinate geometry gives us a way to describe exactly where it is by using two objects. Points are placed on the "region coordinate plane" as shown below in Figure 10. It has two scales: one running across the plane called the "X_A-axis" and another at right angles to it called the "Y_A-axis". Both these axes are thus object linear continuum lines corresponding to the complete region A. The point where the two axes cross is called the origin denoted by O_A at which both x_A and y_A are 0_A. On the X_A-axis, as explained earlier that objects to the right of origin are positive objects and those to the left are negative objects of A. Similarly, on the Y_A-axis, objects above the origin are positive objects and those below the origin are negative objects of A.

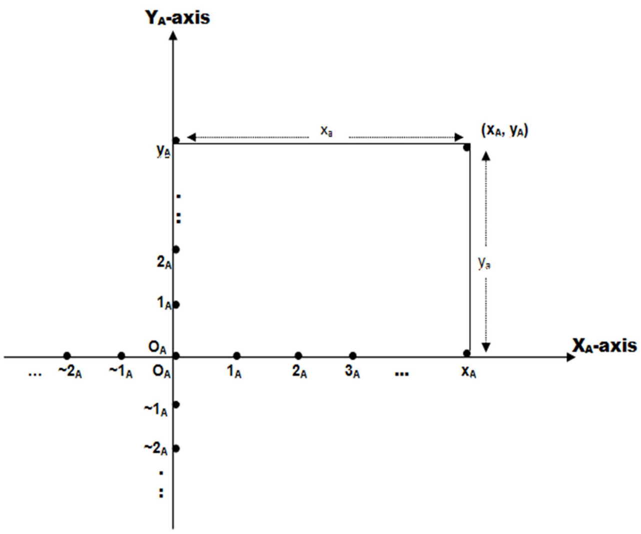


Fig. 10. *Objects coordinates on region coordinate plane of the complete region A.*

A point's location on the region coordinate plane is given by two objects in the form of object coordinates (x_A, y_A), the first coordinate reveals where it is away from the Y_A-axis at parallel to the X_A-axis and the second coordinate reveals where it is away from the X_A-axis at parallel to the Y_A-axis (see Figure 10 above). There are four quadrants and sign convention rule is same as that of classical Cartesian coordinate geometry.

3.2. Slope of an Object Line on the Region Coordinate Plane

Slope of an object line passing through the two object points $P(x_{1A}, y_{1A})$ and $Q(x_{2A}, y_{2A})$ is the real number m_a given by (as shown in Figure 11):

$$m_a = \tan\theta = \frac{y_{2a} - y_{1a}}{x_{2a} - x_{1a}} \quad \text{where } y_{2a} = \| y_{2A} \| \text{ etc.}$$

$$= \frac{y_2 \cdot 1_a - y_1 \cdot 1_a}{x_2 \cdot 1_a - x_1 \cdot 1_a}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

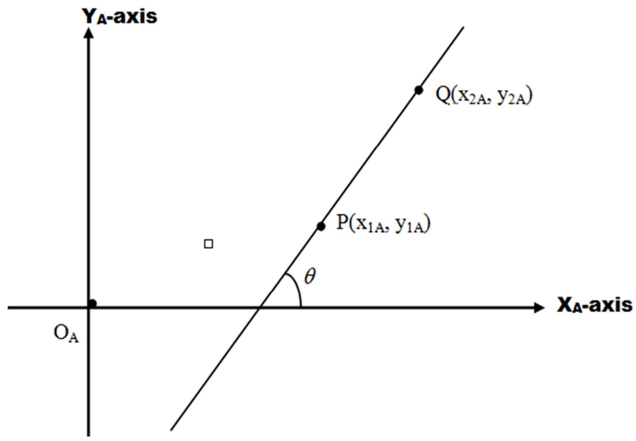


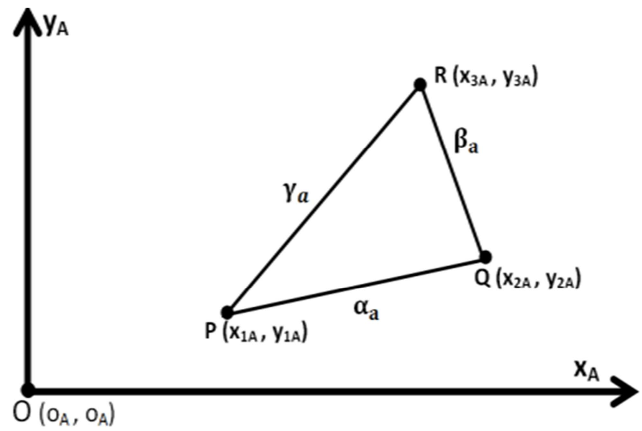
Fig. 11. *Slope of an objects line.*

This implies that slope of a line does not depend on the 'unit length' of the concerned complete region. It is an absolute quantity irrespective of the complete region on which the region coordinate plane is drawn. Thus, slope of a line is region-independent.

Proposition 3.1

Pythagoras Theorem is valid in every Region Geometry, whatever be the corresponding complete region A.

Proof:



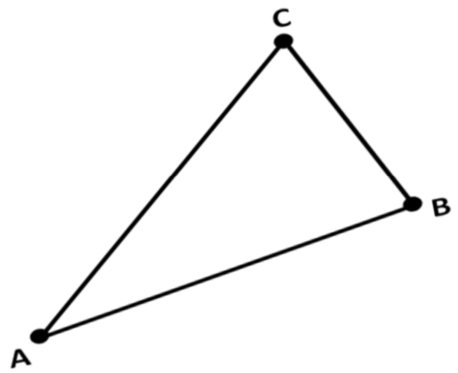


Fig. 12. (a),(b) Two right angled triangles in two region coordinate planes respectively.

Consider the Region Geometry corresponding to the 1-D complete region A. Let PQR be a right angled triangle (the angle PQR being the right angle) on the region coordinate plane of the complete region A (Figure 12(a)). Consider also the 1-D complete region RR. Suppose that the region coordinate plane of A does also represent the region coordinate plane of RR taking same lines as two axes and taking the same location for origin (i.e. O_A and O_{RR} are co-incident points).

Now, using the homogeneity property (see subsection 4.4.1 in [11]) of the metric ρ (x, y) = $\|x{\sim}y\|$, we can find a right-angled triangle ABC on the real Cartesian coordinate plane i.e. on the region coordinate plane of RR region, where

$$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC} = 1_a \quad (1)$$

Since slope of a line is region-independent, the right-angled property of the classical triangle ABC is guaranteed (the angle ABC being the right angle, see Figure 12(b)) on the Cartesian coordinate plane from the right-angled property of the triangle PQR on the region plane of complete region A. Since Pythagoras theorem is valid in the triangle ABC, it is also so in the triangle PQR using result (1). Hence proved.

3.3. Distance Between Two Object Points on a Region Coordinate Plane

Distance between two object points on a region coordinate plane can be defined in various ways like in classical geometry. However, we follow the style of Eucledian distance in Region Geometry. Consider the X_AY_A region coordinate plane corresponding to the complete region A. Let $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ be two points on this region plane (see Figure 13). Distance PQ between these two points in Region Mathematics is the positive real number r_a, where

$$r_a = \left\{ \left(\rho(y_{2A}, y_{1A})\right)^2 + \left(\rho(x_{2A}, x_{1A})\right)^2 \right\}^{\frac{1}{2}}.$$

It may be recollected that for any object x_A we have the relation $x_A = x \bullet 1_A$, the standard notations used in the 'Theory of A-Numbers'. We now see that

$$r_a = \left\{ \|y_{2A} \sim y_{1A}\|^2 + \|x_{2A} \sim x_{1A}\|^2 \right\}^{\frac{1}{2}}$$

$$\text{or, } r_a = \left\{ \|y_2 \bullet 1_A \sim y_1 \bullet 1_A\|^2 + \|x_2 \bullet 1_A \sim x_1 \bullet 1_A\|^2 \right\}^{\frac{1}{2}}$$

$$\text{or, } r_a = \left\{ \|(y_2 - y_1) \bullet 1_A\|^2 + \|(x_2 - x_1) \bullet 1_A\|^2 \right\}^{\frac{1}{2}}$$

$$\text{or, } r_a = \left\{ (y_2 - y_1)^2.\|1_A\|^2 + (x_2 - x_1)^2.\|1_A\|^2 \right\}^{\frac{1}{2}}$$

$$\text{or, } r_a = \left\{ (y_2 - y_1)^2.1_a^2 + (x_2 - x_1)^2.1_a^2 \right\}^{\frac{1}{2}}$$

$$\text{or, } r_a = \left\{ (y_2 - y_1)^2 + (x_2 - x_1)^2 \right\}^{\frac{1}{2}}.1_a \quad (2)$$

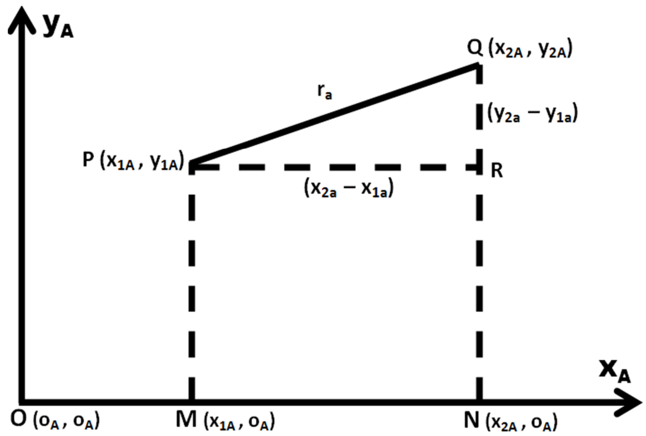


Fig. 13. Distance between two object points.

As a particular case if we take the complete region RR where $1_{rr} = 1$, then the above formula reduces to the classical Eucledian distance formula:

$$r_{rr} = \left\{ (y_2 - y_1)^2 + (x_2 - x_1)^2 \right\}^{\frac{1}{2}}.1_{rr}$$

$$\text{i.e. } r_r = \left\{ (y_2 - y_1)^2 + (x_2 - x_1)^2 \right\}^{\frac{1}{2}} \quad (3)$$

It may be noted that the distance between two points $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ is region dependent and also dependent upon the properties of the chain and the norm of the complete region.

3.4. Equation of an Object Line

Consider the X_AY_A region coordinate plane corresponding to the complete region A (see Figure 14). The general equation of an object line whose slope is m_a is $y_A = m_a \bullet x_A \oplus c_A$.

Equation of an object line having slope m_a and passing through the object point $Q(x_{1A},y_{1A})$ is $(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A})$.

Equation of an object line passing through the two object points $P(x_{1A},y_{1A})$ and $Q(x_{2A},y_{2A})$ is

$$(y_A \sim y_{1A}) = m_a \bullet (x_A \sim x_{1A}), \text{ where } m_a = \frac{y_{2a} - y_{1a}}{x_{2a} - x_{1a}}.$$

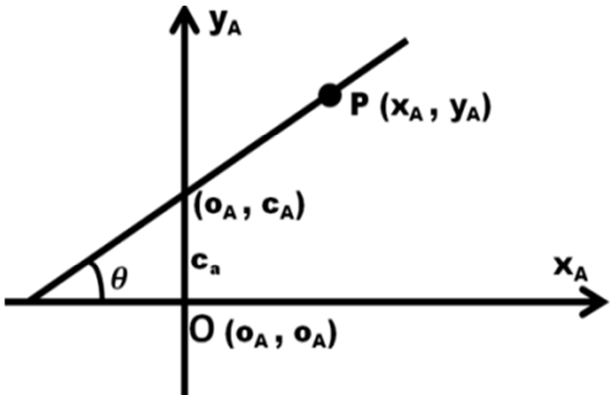


Fig. 14. *An object line having positive intercept of length c_a on Y_A axis.*

3.5. Object Circle on a Region Coordinate Plane

Consider the X_AY_A region coordinate plane corresponding to the complete region A (see Figure 15). Then the equation of an Object Circle with centre at $(0_A, 0_A)$ and radius r_a (>0) is given by

$$\left(\rho\left(x_A,0_A\right)\right)^2 + \left(\rho\left(y_A,0_A\right)\right)^2 = r_a^2$$

which can be written as

$$\|x_A\|^2 + \|y_A\|^2 = r_a^2$$

$$\text{or, } x_a^2+y_a^2 = r_a^2 \qquad (4)$$

$$\text{or, } x^2.1_a^2 + y^2.1_a^2 = r^2.1_a^2$$

$$\text{or, } x^2 + y^2 = r^2 \qquad (5)$$

Thus $x^2 + y^2 = r^2$ represents the equation of the object circle C_1 with radius r_a and centre at $(0_A, 0_A)$ on the region coordinate plane of the region A.

It is to be noted that $x^2 + y^2 = r^2$ does also represent the equation of the object circle C_2 with radius r_b and centre at $(0_B, 0_B)$, but on the region coordinate plane of the region B; and similarly $x^2 + y^2 = r^2$ is also the equation of the object circle C_3 with radius r_{rr} (= r) and centre at $(0_{RR}, 0_{RR})$ i.e. at (0, 0), but on the region coordinate plane of the region RR, etc. Each of these distinct circles C_1, C_2 and C_3 has the common equation $x^2 + y^2 = r^2$ but of different radii (viz. r_a on the region coordinate plane of the complete region A, r_b on the region coordinate plane of the complete region B, and r_{RR} on the region coordinate plane of the complete region RR), as they are on different region coordinate planes. However, the circle C_3 is our classical circle of classical plane geometry. Thus the general equation $x^2 + y^2 = r^2$ of a circle is absolutely region-dependent. If one asks the questions : "What is the radius and centre of the circle $x^2 + y^2 = r^2$? What is the area of it?", then we can not answer immediately unless we know the identity of the concerned complete region i.e. of the concerned region coordinate plane. Consequently, these questions are incomplete questions if asked without mentioning the corresponding complete region. However, in classical geometry, by default it is the complete region RR.

It is interesting to note that in Region Geometry the equation of a circle may be written either using the variables as in equation (4) or using the variables as in equation (5). Both the ways are equivalent in Region Geometry. As a special case both the types of variables happen to be same in classical geometry.

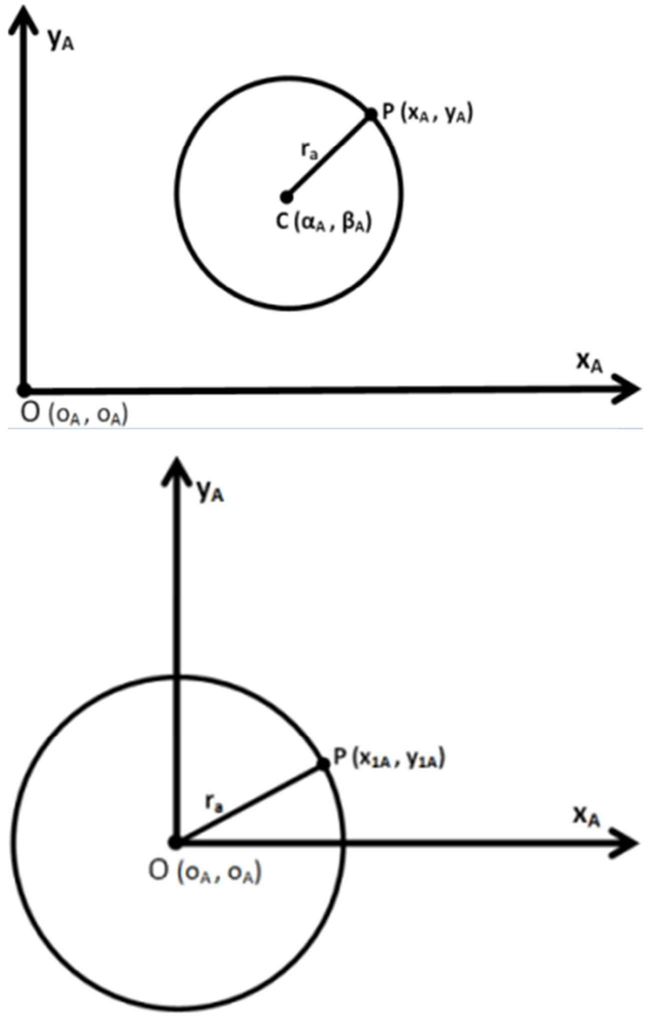


Fig. 15. *(a),(b). Objects circles.*

If 1_a> 1, then the object circle $x^2 + y^2 = r^2$ in the region coordinate plane of the region A is a bigger circle than the classical circle $x^2 + y^2 = r^2$; if 1_a< 1, then the object circle $x^2 + y^2 = r^2$ in the region coordinate plane of the region A is a smaller circle than the classical circle $x^2 + y^2 = r^2$; and if 1_a = 1, then the object circle $x^2 + y^2 = r^2$ is of same size with the classical circle $x^2 + y^2 = r^2$.

Equation of an Object Circle with centre at (α_A, β_A) and radius r_a (>0) is

$$(y_a - \beta_a)^2 + (x_a - \alpha_a)^2 = r_a^2$$

$$\text{or, } (y - \beta)^2 + (x - \alpha)^2 = r^2$$

Thus $(y - \beta)^2 + (x - \alpha)^2 = r^2$ represents the equation of the object circle C_1 (say) with radius r_a and centre at (α_A, β_A) on the region coordinate plane of the complete region A. Again, $(y - \beta)^2 + (x - \alpha)^2 = r^2$ does also represent the equation of the object circle C_2 (say) with radius r_b and centre at (α_B, β_B), but on the region coordinate plane of the complete region B, and similarly $(y - \beta)^2 + (x - \alpha)^2 = r^2$ is also the equation of the object circle C_3 (say) with radius r_{rr} (= r) and centre at $(\alpha_{RR}, \beta_{RR})$ i.e. at (α, β), but on the region coordinate plane of the complete region RR, etc. Each of these distinct circles C_1, C_2 and C_3 has the equation $(y - \beta)^2 + (x - \alpha)^2 = r^2$ but of different radii as they are on different region coordinate planes. However, the circle C_3 is our classical circle of classical plane geometry. Thus the general equation $(y - \beta)^2 + (x - \alpha)^2 = r^2$ of a circle is absolutely region-dependent.

The classical geometry (2-D geometry, 3-D or higher dimensional geometry) being practiced by the world mathematicians at elementary [18] or higher level is a particular case of the 'Region Geometry'. In the "Theory of Objects", the 'Region Geometry' introduced here is just at its infant stage. With a rigorous amount of research work, it will surely take its own shape in future to update the existing classical subject "Geometry".

4. Conclusion

As mentioned in [11] that the work on "Region Mathematics" was not initiated in my mind with any pre-posed problem or plan. I did not have any pre-proposed synopsis for it. It was an accidental development in my mind while I observed that in general the existing standard algebraic systems alone viz. groups, rings, modules, fields, linear spaces, algebra over a field, associative algebra over a field, division algebra, etc. can not validate many of the fundamental and classical equalities, identities, expressions, equations, formulas, results of "elementary algebra" (of secondary school level or higher level) by virtue of their respective definitions and properties. For details about Region Algebra and Region Calculus, one could see the work [11]. The region algebra is applied in the newly discovered "NR-Statistics" [10] in which the population data are not always real numbers but any kind of other real life objects (viz. a population of 50 paints of beautiful TAJMAHAL by 50 junior artists in a school level competition held at Calcutta High school, a collection of 10 X-ray images of a patient during last ten days in Calcutta Medical Hospital, etc. In "NR-Statistics" various new statistical region measures like : region mean, region standard deviation, region variance, etc. with algebraic approach (in the Algebraic Statistics part of NR-Statistics) were studied for real life NR-populations. Philosophically, if we consider the evolution of various algebraic systems, in particular considering their flexible roles and volume of contributing capabilities towards the subjects from 'elementary algebra' to 'higher algebra', we could visualize the unique location of "Region" as mentioned below, which has been unearthed in this work:-

Group → Ring → Field → Linear Space → Division-Algebra → Region.

In Section-2 we introduce another new family member of Region Mathematics called by "Theory of Objects". Although this theory is at its baby stage, but it is initiated here with three giant topics as follows:

1. "Prime Objects" and "Composite Objects" in a Region
2. "Imaginary Objects" and "Compound Objects" in a Region
3. "Theory of Numbers": Every Complete Region has its own

The existing notion of 'prime numbers' is a special case of 'prime objects', and the existing notion of 'composite numbers' is a special case of 'composite objects'. We then define imaginary objects (if exist) of a region. As a particular case we study the existing notion of imaginary number i of the set R of real numbers, as a particular instance of imaginary object, which is called by 'rim'. Another major breakthrough in "Region Mathematics" we unearth is that the region C (set of complex numbers) has at least one imaginary object. Any atomic imaginary object of Cis called by the notation 'cim' of C. One cim we have extracted here which we name by e. If x and y are in R, then corresponding to the rim i of R the object (x+iy) is a complex number but for the set R of real numbers. The object (x+iy) is a complex number locally in the jurisdiction of R. Analogously, if z_1 and z_2 are in C then corresponding to the cim e of C the object $(z_1 + e\, z_2)$ is a compound number for the set C of complex numbers. The rim i is imaginary for R, not for C; and also the rim i is a core member of C, not of R. Thus rim i is a real object of C. The cim e is imaginary for C, not for any other region in general. Being the imaginary object in C, the cim e is not a member of C, i.e. not a real object of C. Thus we have happened to see now the birth of a new type of numbers called by 'compound numbers' of the form $(z_1 + e\, z_2)$ where z_1 and z_2 are in C. All these compound numbers of the form $(z_1 + e\, z_2)$ are so with respect to the cim e, but there could be more than one cim of C of atomic nature (which we can not rule out at this stage). The set of all compound numbers is denoted by E. We need to understand the set E more precisely, by identifying precisely all its members, characteristic properties, results, etc. which will be our future course of research work. In the "Theory of Objects", the set E of Compound Numbers introduced here is just at its infant stage, but undoubtedly it is a new set of numbers discovered here. With a rigorous amount of research work on the set E of numbers, it will surely take its own shape in future to update the existing classical "Theory of Numbers".

For studying prime and composite objects, imaginary objects, etc. we have considered simple regions, not complete regions. But, mathematically there are infinite number of distinct complete regions exist in Region Mathematics. We then introduce another new theory called by "Theory of A-numbers" which is developed if A is a complete region, otherwise not valid. This new giant topic "Theory of A-numbers" will surely grow in a lot of volume with time, the present work being just an initialization. We have identified 'What are the minimum properties which need to be satisfied by a set A so that a new Geometry can be developed over the platform A?'. It has been explained how the classical "Theory of Numbers" being practiced by the world so far happens to be a topic of "Theory of RR-numbers" where the "Theory of RR-numbers" is a particular instance of our new "Theory of A-Numbers" of a complete region A. In every complete region A, there are ontegers ……, ~ 3_A, ~ 2_A, ~ 1_A, 0_A, 1_A, 2_A, 3_A, 4_A, 5_A, ….., and a particular instance of ontegers are the integers ….., -3, -2, -1, 0, 1, 2, 3, 4, ….. which are the ontegers in the complete region RR. Every complete region has its own "unit length", like the unit length in the complete region RR is equal to the distance of the onteger (integer here) 1 from the onteger (integer here) 0.

Consequently, upon the discovery of many more new types of numbers, and by the proposed 'Theory of A-numbers' corresponding to every complete region A, we need to revisit many of the existing famous results [13] in our future work, viz:

(i) R, C, H, O are the only normed division algebras.
(ii) the only associative real division algebras are real numbers, complex numbers, and quaternions.
(iii) The Cayley algebra is the only non-associative division algebra.
(iv) The algebras of real numbers, complex numbers, quaternions, and Cayley numbers are the only ones where multiplication by unit "vectors" is distance-preserving.

In Section-3 we introduce another new giant branch of Region Mathematics called by "Region Geometry". Corresponding to every complete region there is a unique region geometry. The existing 'classical geometry' is one example of the "Region geometry" corresponding to the particular region RR. For a non-example, the set of all triangular fuzzy numbers (or the set of all trapezoidal fuzzy numbers) is closed with respect to the addition operator defined over them, but is not closed with respect to the multiplication operator defined over them [8, 9]. Thus the set of all triangular fuzzy numbers do not form a real region with respect to its commonly used operators (which can not open any platform to develop any calculus), and hence can not open any new Theory of A-Numbers or any new Region Geometry at the present form in "Region Mathematics : A New Direction In Mathematics". The set C of complex numbers does not satisfy the required conditions to become a calculus space with respect to its popular norm $|z| = \sqrt{z\bar{z}}$. Thus no 1-D region calculus can be developed in C, and hence C is not a 1-D complete region with respect to 2-to-1 bijection. Consequently, a number theory of type 'Theory of A-numbers' can not be developed in C, and due to same reason a region geometry too can not be developed in C; although the concept of prime and composite objects, imaginary and complex objects, compound numbers, etc. can be well studied in C. However if C forms a multi-dimensional complete region (say 2-D complete region or n-D) then C may open a new Theory of C-Numbers with multi-dimensional approach (as mentioned in Section-4.8 in [11]) and its own Region Geometry, which is our future course of research work.

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