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On the Finding the Other Eigenvalues and Eigen Functions and Ortogonal Basis with a Nonlocal Parity Condition of the Third Kind

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Abstract: In the present paper, we find out the eigenvalues and the corresponding eigenfunctions of the modified Frankl problem with a nonlocal parity condition, the completeness and the basis property in the elliptic part of the third kind of a domain in $L^2(0,\frac{\pi}{2})$. We also consider a new boundaries condition and analyze the orthogonal basis of the eigenfunctions depending on parameters of the problem.

Keywords: Frankl Problem, Lebesgue Integral, Holder Inequality, Bessel Equation

1. Introduction

The classical Frankl problem was considered in [3]. The problem was further developed in [2, pp.339-345], [12, pp.235-252]. Several authors have also have investigated this issue (see [1-13]). On the solution of the Frankl prolem in a special domain in [12]. About spectrum of the gasedynamic problem of Frankl for the model equation of mixed type in [10]. About construction of the gasedynamic problem of Frankl in [11]. Basis property of eigen -functions of the generalized problem of Frankl with a nonlocal parity condition and with the discontinuity of the gradient of solution in [9].

In the present paper, we consider boundaries conditions of the third kind on the intervals (-1,0) and (0,1) of the axis OY for which the derivatives of functions with respect to x on these intervals are related by linear dependence. We show that if the dependence coefficient exceeds -1 (the coefficient cannot be zero, since, otherwise, the problem will degenerate), then the systems of eigenfunctions of the problem forms a Riesz basis in the elliptic part of the domain.

2. Statement of the Modified Frankl Problem

Definition 1. Find a solution

$$u(x,y)\in C^0(\overline{D_+\cup D_{-1}\cup D_{-2}})\cap C^2(D_{-1})\cap C^2(D_{-2}),$$

of the modified generalized Frankl problem

$$u_{xx}+\operatorname{sgn}(y)u_{yy}+\mu^2\operatorname{sgn}(x+y)u=0. \tag{1}$$

in $D_+\cup D_-$ and the boundary conditions

$$u(1,\theta)=0,\theta\in[0,\frac{\pi}{2}], \tag{2}$$

in the polar coordinate system

$$u(0,y)=0,y\in[-1,1] \tag{3}$$

$$k\frac{\partial u}{\partial x}(0,y)=\frac{\partial u}{\partial x}(0,-y),y\in(0,1),k\neq 0 \text{ is a constant} \tag{4}$$

$$k\lim_{y\to+0} u(x,y) = \lim_{y\to-0} u(x,y), x\in(0,1). \quad (5)$$

here D_+ is the domain in the top half-plane bounded by a circle

$$\gamma = \{(x,y), x^2+y^2=1\}$$

and the segment [0,1] of the axis OY, and $D_- = D_{-1}\cup D_{-2}$ is the domain in the bottom half space, where D_{-1} is bounded by the characteristic $y=x-1$ and $y=-x$ and the segment [0,1] of the axis OX and D_{-2} is bounded by the characteristic $y=x-1$ and $y=-x$ and the segment [-1,0] of the axis OY.

Definition 2. .System $\{x_n\}_{n\in N}\subset X$ is called complete in X if $\overline{L[\{x_n\}_{n\in N}]}=X$.

Definition 3. .System $\{x_n\}_{n\in N}\subset X$ is called minimal in X if $x_k\notin \overline{L[\{x_n\}_{n\in N}]}, \forall k\in N$.

Remark 4. If the system is minimal in $L_p(I)$, then it is also minimal in $L_p(J)$, for $J\supset I$,and if it is complete in $L_p(J)$ for $J\subset I$.

Theorem 5 ([1,5]). The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series.

In the first series, the eigenvalues $\lambda=\mu_{nk}^2$ are found from the equation

$$J_{4n}(\mu_{nk})=0,$$

where $\mu_{nk}, n,k = 1,2,\ldots$, are roots of the Bessel equation (6), $J_\alpha(z)$, is the Bessel function [6,7, Russian translation], and the eigenfunctions are given by the formulas

$$\begin{aligned}
\tilde{u}_{nk}(r,\theta) &= A_{nk}J_{4n}(\mu_{nk}r)\sin(4n(\frac{\pi}{2}-\theta)), \text{ in } D_{+1},\\
\tilde{u}_{nk}(r,\theta) &= kA_{nk}J_{4n}(\mu_{nk}\rho)\sinh(4n\psi), \quad \text{in } D_{-1},\\
\tilde{u}_{nk}(r,\theta) &= kA_{nk}J_{4n}(\mu_{nk}R)\sinh(4n\varphi), \quad \text{in } D_{-2},
\end{aligned} \quad (6)$$

where

$x=r\cos\theta, y=r\sin\theta \quad 0\le\theta\le\frac{\pi}{2}, r^2=x^2+y^2$ in D_+,

$0<\rho<1, -\infty<\psi<0, \rho^2=x^2-y^2$, in D_{-1}, and, $x=R\sinh\varphi, y=-R\cosh\varphi, \quad 0<\varphi<+\infty, R^2=y^2-x^2$ in D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda}=\tilde{\mu}_{nk}^2$ are found from the equation

$$J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk})=0.$$

where $\Delta=\frac{1}{\pi}\arcsin\frac{\kappa}{\sqrt{1+\kappa^2}}, \Delta\in(-\frac{1}{2},\frac{1}{2})\setminus\{0\}$, $\tilde{\alpha}_n=2+4(n-\Delta)\{x_n\}_{n\in N}\subset X$ and the corresponding eigenfunctions have the form

$$\begin{aligned}
\tilde{u}_{nk}(r,\theta) &= \tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}r)\sin(\widetilde{\alpha_n}(\frac{\pi}{2}-\theta)), \text{ in } D_{+1},\\
\tilde{u}_{nk}(r,\theta) &= k\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}\rho)\sinh(\frac{2x}{x^2+1}\cosh(\widetilde{\alpha_n}\psi)\\
&\quad -x\frac{x^2-1}{x^2+1}\sinh(\widetilde{\alpha_n}\psi)), \quad \text{in } D_{-1},\\
\tilde{u}_{nk}(r,\theta) &= k\tilde{A}_{nk}J_{\tilde{\alpha}_n}(\tilde{\mu}_{nk}R)\sinh(\widetilde{\alpha_n}\varphi), \quad \text{in } D_{-2}.
\end{aligned} \quad (7)$$

Theorem 6 ([5]). The function system

$$\{\cos(4n)(\frac{\pi}{2}-\theta)\}_{n=0}^{\infty}, \{\cos 4(n+\Delta)(\frac{\pi}{2}-\theta)\}_{n=1}^{\infty}, \quad (8)$$

is complete and a Riesz basis in $L^2(0,\frac{\pi}{2})$, provided that $\Delta\in(\frac{-1}{4},\frac{1}{2})\cup(\frac{1}{2},\frac{3}{4})$.

3. Main Results

Consider Frankl problem (1)-(5) with the new boundary condition

$$\frac{\partial u}{\partial x}(0,y)=0, y\in(-1,0)\cup(0,1), \quad (9)$$

$$ku(0,y)=u(0,-y), y\in[0,1], ku(0,0^+)=u(0,0^-), \quad (10)$$

$$k\frac{\partial u}{\partial y}(x,0^+)=\frac{\partial u}{\partial y}(x,0^-), -\infty<k<+\infty, 0<x<1. \quad (11)$$

Theorem 7. The eigenvalues and eigenfunctions of problem (1-5) can be written out in two series.

In the first series, the eigenvalues $\lambda=\mu_{nk}^2$ are found from the equation

$$J_{4n+\Delta}(\mu_{nk})=0, \quad (12)$$

where $\mu_{nk}, n,k = 1,2,\ldots$, are roots of the Bessel equation (6), $J_\alpha(z)$, is the Bessel function [4], and the eigenfunctions are given by the formula

$$u_{nk}=\begin{cases} A_{nk}J_{4n}(\mu_{nk}r)\cos(4n)(\frac{\pi}{2}-\theta), & \text{in } D^+;\\ kA_{nk}J_{4n}(\mu_{nk}\rho)\cosh(4n)\psi, & \text{in } D_{-1};\\ kA_{nk}J_{4n}(\mu_{nk}R)\cosh(4n)\varphi, & \text{in } D_{-2}, \end{cases} \quad (13)$$

where

$x = r\cos\theta, y = r\sin\theta$,for $0 \le \theta \le \frac{\pi}{2}, r^2 = x^2 + y^2$ in D_+ , $x = \rho\cosh\psi, y = \rho\sinh\psi$, for, $0 < \rho < 1, -\infty < \psi < 0, \rho^2 = x^2 - y^2$, in D_{-1} , and, $x = R\sinh\varphi, y = -R\cosh\varphi$, for, $0 < \varphi < +\infty, R^2 = y^2 - x^2$ in D_{-2}.

In the second series, the eigenvalues $\tilde{\lambda} = \tilde{\mu}_{nk}^2$ are found from the equation

$$J_{4(n+\triangle)}(\tilde{\mu}_{nk}) = 0. \tag{14}$$

where $n, k = 1, 2, \ldots$ and the $(\tilde{\mu}_{nk})$ are the roots of the Bessel equation (8).

$$u_{nk} = \begin{cases} \tilde{A}_{nk} J_{4(n+\triangle)}(\tilde{\mu}_{nk} r)\cos 4(n+\triangle)(\frac{\pi}{2} - \theta), & \text{in } D^+; \\ \tilde{A}_{nk} J_{4(n+\triangle)}(\tilde{\mu}_{nk}\rho)\cdot \\ \quad [\cosh 4(n+\triangle)\varphi\cos 4(n+\triangle)\frac{\pi}{2} & \text{in } D_{-1}; \\ \quad + \kappa\sinh 4(n+\triangle)\psi\cos 4(n+\triangle)], \\ k\tilde{A}_{nk} J_{4(n+\triangle)}(\tilde{\mu}_{nk} R)\cdot \\ \quad \cosh(4(n+\triangle)\varphi)\cdot & \text{in } D_{-2}, \\ \quad [\cos 4(n+\triangle)\frac{\pi}{2} - \sin 4(n+\triangle)\frac{\pi}{2}], \end{cases}$$

where $\Delta = \frac{1}{\pi}\arcsin\frac{\kappa}{\sqrt{1+\kappa^2}}, \Delta \in (0, \frac{1}{2})$, and

$$A_{nk}^2 \int_0^1 J_{4n}^2(\mu_{nk} r) r dr = 1,$$

$$\tilde{A}_{nk}^2 \int_0^1 J_{4n+\triangle}^2(\tilde{\mu}_{nk} r) r dr = 1,$$

$$A_{nk} > 0 \text{ and } \tilde{A}_{nk} > 0.$$

Theorem 8. The function system

$$\{\cos(4n)(\frac{\pi}{2} - \theta)\}_{n=0}^{\infty}, \{\cos 4(n+\triangle)(\frac{\pi}{2} - \theta)\}_{n=1}^{\infty}, \tag{15}$$

is complete and a Riesz basis in $L^2(0, \frac{\pi}{2})$, provided that $\triangle \in (\frac{-1}{4}, \frac{1}{2})$.

Proof. In order to prove this theorem we use the method in [1, 6] by considering convergence function

$$f(\theta) = \sum_{n=0}^{\infty} A_n \cos 4n(\frac{\pi}{2} - \theta) + \sum_{n=1}^{\infty} B_n \cos 4(n+\triangle)(\frac{\pi}{2} - \theta),$$

in $L^2(0, \frac{\pi}{2})$ and Riesz basis the system $(\sin 4(n+\triangle)(\frac{\pi}{2} - \theta))$ for $\triangle \in (\frac{-1}{4}, \frac{3}{4})$.

Remark 9. For $\triangle < \frac{-1}{4}$ the system (10) is not complete but is minimal, for $\triangle > \frac{3}{4}$ is complete but is not minimal, and if $\triangle = \frac{-1}{4}$, is complete and minimal.

Theorem 10. The system of eigenfunctions

$$u_{nk}(r,\theta) = A_{nk} J_{4n}(\mu_{nk} r)\cos(4n)(\frac{\pi}{2} - \theta),$$

$$\tilde{u}_{nk}(r,\theta) = \tilde{A}_{nk} J_{4(n+\triangle)}(\tilde{\mu}_{nk} r)[\cosh 4(n+\triangle)\varphi\cos 4(n+\triangle)\frac{\pi}{2}],$$

is complete and basis in the space $L^2(0, \frac{\pi}{2})$, therefore

$$\int_0^{\frac{\pi}{2}} f(r,\theta) u_{nk}(r,\theta) r dr d\theta = 0,$$

$$\int_0^{\frac{\pi}{2}} f(r,\theta)\tilde{u}_{nk}(r,\theta) r dr d\theta = 0,$$

and $f \in L^2(0, \frac{\pi}{2})$ then $f = 0$ in $(0, \frac{\pi}{2})$.

Proof. Using Fubini theorem for any $n, k = 1, 2, \ldots$ we have

$$0 = \int_0^{\frac{\pi}{2}} f(r,\theta) u_{nk}(r,\theta) r d\theta dr$$

$$\int_0^1 (r J_{4n}(\mu_{nk} r)\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2} - \theta) d\theta) dr,$$

again since $f \in L^2\left(0, \frac{\pi}{2}\right)$ so;

$$\int_0^1 \int_0^{\frac{\pi}{2}} |f(r,\theta)|^2 \, d\theta dr < \infty.$$

and since the system $\{\sqrt{r} J_{4n}(\mu_{nk} r)\}_{k=1}^{\infty}$ in $L^2(0,1)$ is orthogonal and complete, it is enough to prove:

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2} - \theta) d\theta \in L^2(0,1).$$

Using the Holder inequality

$$|\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2 < \frac{1}{2}r\int_0^{\frac{\pi}{2}}|f^2(r,\theta)|\,d\theta\int_0^{\frac{\pi}{2}}d\theta$$

$$=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2\,d\theta=\frac{\pi}{4}r\int_0^{\frac{\pi}{2}}|f(r,\theta)|^2\,d\theta,$$

with the integration interval (0,1).

$$\int_0^1|\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2\,dr?$$

$$<\frac{\pi}{4}\int_0^1\int_0^{\frac{\pi}{2}} r\,|f(r,\theta)|^2\,drd\theta<\infty.$$

This inequality is equivalent to

$$\{\int_0^1\sqrt{r}\,|\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta|^2\,dr\}^{\frac{1}{2}}<\infty.$$

Also system $\{\sqrt{r}J_{4n}(\mu_{nk}r)\}_{k=1}^{\infty}$ is orthogonal and complete in $L^2(0,\frac{\pi}{2})$ of relation

$$\int_0^1(\sqrt{r}J_{4n}(\mu_{nk}r)\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta)dr=0,$$

imply that

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos(4n)(\frac{\pi}{2}-\theta)d\theta=0.$$

According to theorem 6, we conclude that $f(r,\theta)=0$ in $L^2(0,1)$. Similarly, if we consider the above calculations for sequence $\{\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)\}_{n=1}^{\infty}$, we have

$$\sqrt{r}\int_0^{\frac{\pi}{2}} f(r,\theta)\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)d\theta=0.$$

Because completeness $\{\cos 4(n+\triangle)(\frac{\pi}{2}-\theta)\}_{n=0}^{\infty}, f(r,\theta)=0$ in $L^2(0,1)$.

The proof of the theorem is complete.

Remark 11. If $\Delta=0$ then the system becomes the system $\{\sin(2n\theta)\}_{n=1}^{\infty}$ which is basis in the space $L^p(0,\frac{\pi}{2})$ and an orthogonal basis in the space $L^2(0,\frac{\pi}{2})$.

The proof of remark 11 results from theorem 10.

Remark 12. In case $\triangle>\frac{3}{4}$ and $\Delta\neq\frac{1}{2}+k, k\in\mathbb{N}$ then the system is complete but is not minimal.

In case $\triangle<\frac{-1}{4}$ and $\Delta\neq-\frac{1}{4}-k, k\in\mathbb{N}$ then the system is not complete but is minimal.

In case $\triangle=\frac{1}{2}+k, k\in\mathbb{Z}$ then the system is complete but is not minimal in the space $L^2(0,\frac{\pi}{4})$ and is not complete in the space $L^2(0,\frac{\pi}{2})$.

The proof of remark 12 results from theorem 8.

4. Conclusion

Consider Frankl problem (1)-(5) with the new boundary condition (9)-(10) and (11), so we find out the eigenvalues of the problem with a nonlocal parity condition ,the completeness and the basis property in the elliptic part of the third kind of a domin in $L^2(0,\frac{\pi}{2})$.

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