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Common Fixed-Point Theorems in G-complete Fuzzy Metric Spaces

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Abstract: Following the approach of Gregori and Sapena, in this paper we introduced a new class of contractions and we establish some common fixed point theorems in G-complete fuzzy metric. Also a theorem on the equivalency related to completeness is given. The results are a genuine generalization of the corresponding results of Gregori and Sapena.

Keywords: G-complete, Fuzzy Metric Spaces, Common Fixed Point

1. Introduction

Several authors [1, 4, 5, 10] have proved fixed point theorems for contractions in fuzzy metric spaces, using one of the two di erent types of completeness: in the sense of Grabiec [4], or in the sense of Schweizer and Sklar [3, 9]. Gregori and Sapena [5] introduced a new class of fuzzy contraction mappings and proved several fixed point theorems in fuzzy metric spaces. Gregori and Sapena's results extend classical Banach fixed point theorem and can be considered as a fuzzy version of Banach contraction theorem. In this paper, following the results of [5] we give a new common fixed point theorem in the two different types of completeness and by using the recent definition of contractive mapping of Gregori and Sapena [5] in fuzzy metric spaces.

Recall [9] that a continuous t-norm is a binary operation $*:[0,1]\times[0,1]\to[0,1]$ such that $([0,1],\leq,*)$ is an ordered Abelian topological monoid with unit 1. The two important t-norms, the minimum and the usual product, will be denoted by min and $\cdot$, respectively.

Definition 1.1 ([3]). A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set of $X\times X\times(0,1)$ satisfying the following conditions, for all $x,y,z\in X, s,\ t>0$:

(FM1) $M(x,y,t)>0$;

(FM2) $M(x,y,t)=1$ if and only if $x=y$;

(FM3) $M(x,y,t)\ = M(y,x,t)$;

(FM4) $M(x,z,t+s)\geq M(x,y,t)*M(y,z,\ s)$;

(FM5) $M(x,y,\cdot):(0,+\infty)\to[0,1]$ is continuous.

If, in the above definition, the triangular inequality (FM4) is replaced by (NAF)

$$M(x,z,t)\geq M(x,y,t)*M(y,\ z,\ t),\forall x,y,z\in X,\forall t>0,$$

then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space.

Remark 1.2 ([3]). In fuzzy metric space $(X,M,*)$, $M(x,y,\cdot)$ is non-decreasing for all $x,y\in X$.

Definition 1.3 ([4]). A sequence x_n in X is said to be convergent to a point x in X (denoted by $x_n\to x$), if $M(x_n,x,t)\to 1$,for all $t>0$.

Definition 1.4 ([3,5]). Let be $(X,M,*)$ a fuzzy metric space.

a) A sequence x_n is called G-Cauchy if for each $t>0$ and $p\in\mathbb{N}$, $\lim_{n\to\infty}M(x_{n+p},x_n,t)=1$. The fuzzy metric space $(X,M,*)$ is called G-complete if every G-Cauchy sequence is convergent.

b) A sequence x_n is called Cauchy sequence if for each $\epsilon\in(0,1)$ and each $t>0$ there exists $n_0\in\mathbb{N}$ such that or all $n,m\geq n_0$. The fuzzy metric space $(X,M,*)$ is called complete if every Cauchy sequence is convergent.

Remark 1.5 ([7]). Let $(X, M, *)$ be a fuzzy metric space then M is a continuous function on $X \times X \times (0,1)$.

2. Main Results

In this section, we extend common fixed point theorem of generalized contraction mapping in fuzzy metric spaces, our work is closely related to [1,2, 5]. Gregori and Sepena introduced the notions of fuzzy contraction mapping and fuzzy contraction sequence as follows:

Definition 2.1 ([5]). Let be $(X, M, *)$ a fuzzy metric space.

a) We call the mapping $T : X \to X$ is fuzzy contractive mapping, if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(Tx, Ty, t)} - 1 \le \lambda\left(\frac{1}{M(x, y, t)} - 1\right)$$

for each $x, y \in X$ and $t > 0$.

b) Let $(X, M, *)$ be a fuzzy metric space. A sequence x_n is called fuzzy contractive if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(x_n, x_{n+1}, t)} - 1 \le \lambda\left(\frac{1}{M(x_{n-1}, x_n, t)} - 1\right)$$

for every $t > 0, n \in \mathbb{N}$.

Theorem 2.2. Let $(X, M, *)$ be a G-complete fuzzy metric space endowed with minimum t-norm and $\{\mathrm{T}_\alpha\}_{\alpha \in \mathrm{J}}$ be a family of self-mappings of X. If there exists a fixed $\beta \in \mathrm{J}$ such that for each $\alpha \in \mathrm{J}$

$$\frac{1}{M(T_\alpha x, T_\beta y, t)} - 1 \le \lambda \max\left\{\left(\frac{1}{M(x, y, t)} - 1\right), \left(\frac{1}{M(x, T_\alpha x, t)} - 1\right), \left(\frac{1}{M(y, T_\beta y, t)} - 1\right), \left(\frac{1}{M(x, T_\beta y, t)} - 1\right), \left(\frac{1}{M(y, T_\alpha x, 2t)} - 1\right)\right\} \quad (2.1)$$

for some $\lambda = \lambda(\alpha)$ and for each $x, y \in X, t > 0$. Then all T_α have a unique common fixed point and at this point each T_α is continuous.

Proof. Let $\alpha \in \mathrm{J}$, $\mathrm{x} \in \mathrm{X}$ and $t > 0$ be arbitrary. Consider a sequence, defined inductively $x_0 = x,\ x_{2n+1} = T_\alpha(x_{2n}), x_{2n+2} = T_\alpha(x_{2n+1})$ for all $n \ge 0$. From (2.1) we get

$$\begin{aligned} \frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 &= \frac{1}{M(T_\alpha x_{2n}, T_\beta x_{2n+1}, t)} - 1 \\ &\le \lambda \max\left\{\left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right), \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right), \left(\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1\right), \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right), \left(\frac{1}{M(x_{2n}, x_{2n+2}, 2t)} - 1\right)\right\}. \end{aligned} \quad (2.2)$$

Since

$$\frac{1}{M(x_{2n}, x_{2n+2}, 2t)} - 1 \le \frac{1}{\min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+1}, t)\}} - 1 = \max\left\{\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1, \frac{1}{M(x_{2n+1}, x_{2n+1}, t)} - 1\right\}, \quad (2.3)$$

we have

$$\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 \le \lambda \max\left\{\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1, \frac{1}{M(x_{2n+1}, x_{2n+1}, t)} - 1\right\}.$$

hence, as $\lambda < 1$ we get

$$\frac{1}{M(x_{2n+1}, x_{2n+2}, t)} - 1 \le \lambda \max\left\{\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right\}.$$

Similarly, we get that

$$\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1 \le \lambda \max\left\{\frac{1}{M(x_{2n-1}, x_{2n}, t)} - 1\right\}.$$

So $\{x_n\}$ is fuzzy contractive, thus, by proposition [5, Proposition 2.4] is G-Cauchy. Since X is G-complete, $\{x_n\}$ converges to . u . for some $u \in X$. From (2.1) we have

$$\frac{1}{M(T_\beta u, x_{2n+1}, t)} - 1 = \frac{1}{M_\beta(T_\beta u, T_\alpha x_{2n}, t)} - 1$$
$$\leq \lambda \max\left\{\left(\frac{1}{M(u, x_{2n}, t)} - 1\right), \left(\frac{1}{M(u, T_\beta u, t)} - 1\right), \left(\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1\right), \left(\frac{1}{M(u, x_{2n+1}, 2t)} - 1\right), \left(\frac{1}{M(x_{2n}, T_\beta u, 2t)} - 1\right)\right\}.$$

Taking the limit as infinity we obtain

$$\frac{1}{M(T_\beta u, u, t)} - 1 \leq \lambda\left(\frac{1}{M(u, T_\beta u, t)} - 1\right).$$

Thus $M(u, Tu, t) = 1$, hence, $Tu = u$.

Now we show that u is a fixed point of all $\{T_\alpha\}_{\alpha \in J}$. Let α ∈ J. From (2.1) we have

$$\frac{1}{M(u, T_\alpha u, t)} - 1 = \frac{1}{M(T_\beta u, T_\alpha u, t)} - 1 \leq \lambda(\alpha) \max\left\{\left(\frac{1}{M(u, T_\alpha u, t)} - 1\right), \left(\frac{1}{M(u, T_\alpha u, 2t)} - 1\right)\right\}.$$

Hence $T_\alpha u = u$, since α is arbitrary all $\{T_\alpha\}_{\alpha \in J}$ have a common point.

Suppose that v is also a fixed point of T_β . Similarly, as above, v is a common fixed point of all $\{T_\alpha\}_{\alpha \in J}$. Form (2.1) we get

$$\frac{1}{M(v, u, t)} - 1 = \frac{1}{M(T_\beta v, T_\alpha u, t)} - 1 \leq \lambda \max\left(\frac{1}{M(u, T_\alpha u, t)} - 1\right).$$

Thus u is a unique common fixed point of all $\{T_\alpha\}_{\alpha \in J}$. It remains to show each T_α is continuous at u . Let y_n be a sequence in X such that $y_n \to u$ as $n \to \infty$. From (2.1) we have

$$\frac{1}{M(T_\alpha y_n, T_\alpha u, t)} - 1 = \frac{1}{M(T_\alpha y_n, T_\beta u, t)} - 1 \leq \lambda \max\left\{\left(\frac{1}{M(y_n, u, t)} - 1\right), \left(\frac{1}{M(y_n, T_\alpha y_n, t)} - 1\right), 0, \left(\frac{1}{M(y_n, u, t)} - 1\right), \left(\frac{1}{M(u, T_\alpha y_n, 2t)} - 1\right)\right\}$$

and by

$$\frac{1}{M(u, T_\alpha y_n, 2t)} - 1 \leq \frac{1}{M(u, T_\alpha y_n, t)} - 1,$$

we deduce

$$\frac{1}{M(T_\alpha y_n, T_\alpha u, t)} - 1 \leq \frac{\lambda}{1-\lambda}\left(\frac{1}{M(y_n, u, t)} - 1\right).$$

So $M(T_\alpha y_n, \mathrm{T}_\alpha u, t) \to 1$, as $n \to \infty$, for all $t > 0$. Thus T_α is continuous at a fixed point.

Theorem 2.3. Let $(X, M, *)$ be a G-complete fuzzy metric space endowed with minimum t-norm. The following property is equivalent to completeness of X :

If Y is any non-empty closed subset of X and $T : Y \to Y$ is any generalized contraction mapping then T has a fixed point in Y .

Proof. The sufficient condition follows from Theorem 2.2. Suppose now that the property holds, but $(X, M, *)$ is not complete. Then there exists a Chuchy sequence $\{x_n\}$ in X which does not converge. We may assume that $M(x_n, x_m, t) < 1$ for all $m \neq n$ and for some $t > 0$. For any $x \in X$ define

$$r(x) = \inf\{\frac{1}{M(x_n, x, t)} - 1; x_n \neq x, n = 0,\ 1, \ldots\}.$$

Clearly for all $x \in X$ we have $r(x) < 1$, as $\{x_n\}$ has not a convergent subsequence. Let $0 < \lambda < 1$. We choose a subsequence x_{i_n} of $\{x_n\}$ as follows. We define inductively a subsequence of positive integer greater than i_{n-1} and such that

$$\frac{1}{M(x_i, x_k, t)} - 1 \leq \lambda r(x_{i_{n-1}})$$

for all $i, k \geq i_n, n \geq 1$. This can done, as $\{x_n\}$ is a Chuchy sequence.

Now define $Tx_{i_n} = x_{i_{n+1}}$ for all n. Then for any $m > n \geq 0$ we have

$$\frac{1}{M(Tx_{i_n}, Tx_{i_m}, t)} - 1 = \frac{1}{M(x_{i_{n+1}}, x_{i_{m+1}}, t)} - 1 \leq \lambda r(x_{i_m}) \leq \lambda\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right) \leq \lambda \max\left\{\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right),\right.$$

$$\left.\left(\frac{1}{M(x_{i_n}, x_{i_{n+1}}, t)} - 1\right), \left(\frac{1}{M(x_{i_m}, x_{i_{m+1}}, t)} - 1\right), \left(\frac{1}{M(x_{i_n}, x_{i_{m+1}}, t)} - 1\right), \left(\frac{1}{M(x_{i_m}, x_{i_{n+1}}, 2t)} - 1\right)\right\}$$

$$= \lambda \max\left\{\left(\frac{1}{M(x_{i_n}, x_{i_m}, t)} - 1\right), \left(\frac{1}{M(x_{i_n}, Tx_{i_n}, t)} - 1\right), \left(\frac{1}{M(x_{i_m}, Tx_{i_m}, t)} - 1\right), \left(\frac{1}{M(x_{i_n}, Tx_{i_m}, t)} - 1\right), \left(\frac{1}{M(x_{i_m}, Tx_{i_n}, 2t)} - 1\right)\right\}$$

Thus T is a general contraction mapping on $Y = \{x_n\}$. Clearly, Y is closed and T has not a fixed point in T. Thus we get the contraction.

Theorem 2.4. Let $(X, M, *)$ be a complete non-Archimedean fuzzy metric space endowed with minimum t-norm and $\{\mathrm{T}_\alpha\}_{\alpha \in \mathrm{J}}$ be a family of self-mappings of X. If there exists a fixed $\beta \in \mathrm{J}$ such that for each $\alpha \in \mathrm{J}$

$$\frac{1}{M(T_\alpha x, T_\beta y, t)} - 1 \leq \lambda \max\left\{\left(\frac{1}{M(x, y, t)} - 1\right), \left(\frac{1}{M(x, T_\alpha x, t)} - 1\right), \left(\frac{1}{M(y, T_\beta y, t)} - 1\right), \left(\frac{1}{M(x, T_\beta y, t)} - 1\right), \left(\frac{1}{M(y, T_\alpha x, t)} - 1\right)\right\},$$

for some $\lambda = \lambda(\alpha)$ and for each $x, y \in X, t > 0$. Then all T_α have a unique common fixed point and at this point each T_α is continuous.

Proof. The proof is very similar to the Theorem 2.2. Instead of the equation (2.3) we have

$$\frac{1}{M(x_{2n}, x_{2n+2}, t)} - 1 \leq \frac{1}{\min\{M(x_{2n}, x_{2n+1}, t), M(x_{2n+1}, x_{2n+1}, t)\}} - 1 = \max\left\{\frac{1}{M(x_{2n}, x_{2n+1}, t)} - 1, \frac{1}{M(x_{2n+1}, x_{2n+1}, t)} - 1\right\}$$

Proceed as the proof of the Theorem 2.2 then we conclude sequence $\{x_n\}$ is fuzzy contractive, thus by [5, Proposition 2.4] and [6, Lemma 2.5], $\{x_n\}$ converges to u for some $u \in X$. Proceed as the proof of the Theorem 2.2.

3. Conclusion

Motivated by a celebrated result of Gregori and Sapena we introduced a new class of contractions having common fixed points on every G-complete fuzzy metric space $(X, M, *)$ and then we prove a new result for completeness.

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