Pure and Applied Mathematics Journal
2013; 2(2) : 101-105
Published online May 20, 2013 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20130202.18

Asymptotic method for certain over-damped nonlinear vibrating systems

Pinakee Dey

Department of Mathematics, Mawlana Bhashani Science and Technology University, Santosh, Tangail-1902, Bangladesh

Email address:
pinakee_68@yahoo.com

To cite this article:
Pinakee Dey. Asymptotic Method for Certain over-Damped Nonlinear Vibrating Systems. *Pure and Applied Mathematics Journal*. Vol. 2, No. 2, 2013, pp. 101-105. doi: 10.11648/j.pamj.20130202.18

Abstract: Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended and applied to certain over-damped nonlinear system in which the linear equation has two almost equal roots. The method is illustrated by an example.

Keywords: Nonlinear System, Unperturbed Equation, Over-Damped Oscillatory System, Equal Roots

1. Introduction

Krylov-Bogoliubov-Mitroplshkii (KBM) [1-3] method is widely used techniques to obtain approximate solutions of weakly nonlinear system. The method was originally developed for approximating periodic solutions of second order nonlinear differential systems was letter extended by Popov [4] to damped oscillatory nonlinear systems. Murty, Dekshatulu and Krisna [5] extended the method to over-damped nonlinear system. Recently Shamsul [6] has presented a unified method for solving an *n*-th order differential equation (autonomous) characterized by oscillatory, damped oscillatory and non-oscillatory processes with constant coefficients. In another recent article, Shamsul [7],Pinakee, *et,al* [7,8] investigated over-damped nonlinear systems and found approximate solutions of *Duffing*'s equation when one root of the unperturbed equation was double of the other. The aim of this article is to find an approximate solution of over-damped nonlinear differential systems based on the extended KBM (by Popov [4]) method in which one of the eigen-values is almost equal to the other eigen-value.

2. Materials and Method

Consider a nonlinear system governed by the differential equation,

$$\ddot{x} + k_1\dot{x} + k_2 x = -\varepsilon\, f(x,\dot{x}), \tag{1}$$

Where the over-dots denote differentiation with respect to *t*, k_1 and k_2 are constants, ε is a small parameter, f is the given nonlinear function. When $\varepsilon = 0$ (1) has two roots, say λ_1 and λ_2. Therefore, the solution of the unperturbed equation of (1) become

$$x(t,0) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right), \tag{2}$$

Where a_0 and b_0 are arbitrary constant. We choose an approximate solution of (1) in the form of the asymptotic expansion

$$x(t,\varepsilon) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + \varepsilon u_1(a,b,t) + \varepsilon^2 \ldots, \tag{3}$$

Where *a* and *b* satisfy the differential equations

$$\begin{aligned} \dot{a} &= \varepsilon A_1(a,b,t) + \varepsilon^2 A_2(a,b,t) + \varepsilon^3 \ldots, \\ \dot{b} &= \varepsilon B_1(a,b,t) + \varepsilon^2 B_2(a,b,t) + \varepsilon^3 \ldots., \end{aligned} \tag{4}$$

Herein solution (3) together with (4) is not considered in a usual form of the classical KBM method. But this solution was early introduced by Murty and Deekshatulu [5] to investigate an over-damped case of equation (1). Now it is being used to investigate various oscillatory and non-oscillatory problems (see [6-9] for details).

Differentiating $x(t,\varepsilon)$ twice with respect to *t*, substituting the derivatives, $\dot{x}, \ddot{x}$ and $x(t,\varepsilon)$ in the original equation (1) and equating the coefficient of ε, we obtain

$$\frac{1}{2}\left(\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_1 t}+\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_2 t}\right)+\frac{\partial B_1}{\partial t}\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$$
$$(e^{\lambda_1 t}+e^{\lambda_2 t})B_1+\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1=-f^{(0)}(a,b,t) \tag{5}$$

where $f^{(0)}=f(x_0,\dot{x}_0)$ and

$$x(t,0)=\frac{1}{2}a_0(e^{\lambda_1 t}+e^{\lambda_2 t})+b_0\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$$

In general $f^{(0)}$ be expanded in power of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$ as;

$$f^{(0)}=g_0(a,b,t)+g_1(a,b,t)\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}+g_2(a,b,t)\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^2+\ldots \tag{6}$$

Substitute the expansions of $f^{(0)}$ from (6) into (5) and equation the coefficients of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^0,\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^1$ and higher order terms of $\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$, we obtain

$$\frac{1}{2}\left(\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_1 t}+\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_2 t}\right)+$$
$$(e^{\lambda_1 t}+e^{\lambda_2 t})B_1=-g_0(a,b,t) \tag{7}$$

$$\frac{\partial B_1}{\partial t}=-g_1(a,b,t)\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}, \tag{8}$$

and

$$\left(\frac{\partial}{\partial t}-\lambda_1\right)\left(\frac{\partial}{\partial t}-\lambda_2\right)u_1=-g_2(a,b,t)\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^2 - \tag{9}$$

The particular solution of (7)-(9) gives three unknown functions A_1, B_2 and u_1, which complete the determination of the first order solution of (1).

Example: Let us consider a *Duffing* equation with a large linear damping,

$$\ddot{x}+2k\dot{x}+\omega^2 x=-\varepsilon x^3, \tag{10}$$

Here $\lambda_1+\lambda_2=-2k$ and $\lambda_1\lambda_2=\omega^2$.

The function $f^{(0)}$ becomes,

$$f^{(0)}=\frac{a^3}{8}(e^{\lambda_1 t}+e^{\lambda_2 t})^3+\frac{3a^2b}{4}(e^{\lambda_1 t}+e^{\lambda_2 t})^2\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)$$
$$+\frac{3ab^2}{4}(e^{\lambda_1 t}+e^{\lambda_2 t})\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^2+b^3\left(\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}\right)^3$$

Therefore, the nonzero coefficients of g_r $r=0,1,2\ldots$ are $g_0=\frac{a^3}{8}(e^{\lambda_1 t}+e^{\lambda_2 t})^3$, $g_1=\frac{3a^2b}{4}(e^{\lambda_1 t}+e^{\lambda_2 t})^2$, $g_2=\frac{3ab^2}{4}(e^{\lambda_1 t}+e^{\lambda_2 t})$, $g_3=b^3$. Substituting the values of g_1 in to (8) and the values of g_2, g_3 into (9) and solving them, we obtain

$$B_1=\frac{-3a^2b}{4}\left(\frac{e^{2\lambda_1 t}}{2\lambda_1}+\frac{2e^{(\lambda_1+\lambda_2)}}{\lambda_1+\lambda_2}+\frac{e^{2\lambda_2 t}}{2\lambda_2}\right) \tag{11}$$

and

$$u_1=-\frac{3ab^2}{4}\left(\frac{e^{3\lambda_1 t}}{\lambda_1(3\lambda_1-\lambda_2)}-\frac{e^{(2\lambda_1+\lambda_2)t}}{\lambda_1(\lambda_1+\lambda_2)}-\frac{e^{(\lambda_1+2\lambda_2)t}}{\lambda_2(\lambda_1+\lambda_2)}+\frac{e^{3\lambda_2 t}}{\lambda_2(3\lambda_2-\lambda_1)}\right)$$
$$-\frac{b^3}{2(\lambda_1-\lambda_2)^3}\left(\frac{e^{3\lambda_1 t}}{\lambda_1(3\lambda_1-\lambda_2)}-\frac{3e^{(2\lambda_1+\lambda_2)t}}{\lambda_1(\lambda_1+\lambda_2)}+\frac{3e^{(\lambda_1+2\lambda_2)t}}{\lambda_2(\lambda_1+\lambda_2)}-\frac{e^{3\lambda_2 t}}{\lambda_2(3\lambda_2-\lambda_1)}\right) \tag{12}$$

Now substituting the values of g_0 and the values of B_1 from (11) into (7) and simplifying, we obtain

$$\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_1 t}+\left(\frac{\partial A_1}{\partial t}+(\lambda_1-\lambda_2)A_1\right)e^{\lambda_2 t}$$
$$=-\frac{a^2}{4}\left(a-\frac{6b}{\lambda_1+\lambda_2}\right)(e^{\lambda_1 t}+e^{\lambda_2 t})^3-\frac{(\lambda_1-\lambda_2)a^2b}{\lambda_1\lambda_2}(\lambda_2 e^{2\lambda_1 t}-\lambda_1 e^{2\lambda_2 t})(e^{\lambda_1 t}+e^{\lambda_2 t}) \tag{13}$$

It is noted that (13) has not always an exact solution. It has an exact solution when $\lambda_1 = \lambda_2$. However, we can find an approximate solution of (13) when $(\lambda_1 - \lambda_2)^2 \le \varepsilon$. We neglect the last term of (13), since $\lambda_2 e^{2\lambda_1 t} - \lambda_1 e^{2\lambda_2 t}$ is order of $\lambda_1 - \lambda_2$. Therefore, we rewrite (13) as:

$$\left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_1 t} + \left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_2 t} = -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)(e^{\lambda_1 t} + e^{\lambda_2 t})^3 \tag{14}$$

Equation (14) has again not an exact solution unless $\lambda_1 = \lambda_2$. Now we can start with the following equation and a trial solution as:

$$\left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_1 t} + \left(\frac{\partial A_1}{\partial t} + (\lambda_1 - \lambda_2)A_1\right)e^{\lambda_2 t}$$
$$= -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)(e^{3\lambda_1 t} + he^{(2\lambda_1 + \lambda_2)t} + he^{(\lambda_1 + 2\lambda_2)t} + e^{3\lambda_2 t}) \tag{15}$$

and $A_1 = l_1 e^{2\lambda_1 t} + l_2 e^{(\lambda_1 + \lambda_2)t} + l_3 e^{2\lambda_2 t}$ where l_1, l_2, l_3 and h are unknown. Substituting A_1 into (15) and equating the coefficients of $e^{3\lambda_1 t}, \ldots e^{3\lambda_2 t}$, we obtain a set of algebraic equations, whose solutions are

$$l_1 = -\frac{a^2}{4(3\lambda_1 - \lambda_2)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right),$$
$$l_3 = -\frac{a^2}{4(3\lambda_2 - \lambda_1)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)$$
$$l_2 = -\frac{a^2}{4}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right) * \frac{\lambda_1 + \lambda_2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)},$$
$$h = \frac{3(\lambda_1 + \lambda_2)^2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} \tag{16}$$

Now we should investigate the value of h. It is obvious that as $\lambda_1 \to \lambda_2$, $h \to 3$ .When $\lambda_1 \ne \lambda_2$, we obtain

$$3 - h = \frac{-12(\lambda_1 - \lambda_2)^2}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} \tag{17}$$

Therefore, we neglect the right hand side of (17) when $(\lambda_1 - \lambda_2)^2 \le \varepsilon$. Thus an approximate solution of (14) or (13) is

$$A_1 \cong -\frac{a^2}{4(3\lambda_1 - \lambda_2)}\left(a - \frac{6b}{\lambda_1 + \lambda_2}\right)\left(\frac{e^{2\lambda_1 t}}{(3\lambda_1 - \lambda_2)} + \frac{2(\lambda_1 + \lambda_2)e^{(2\lambda_1 + \lambda_2)t}}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} + \frac{e^{2\lambda_2 t}}{(3\lambda_2 + \lambda_1)}\right) \tag{18}$$

Now substituting the values of A_1 from (18) and B_1 from (11) into (4) and then integrating with respect to t by assuming that a and b are constant in the right sides of (4), we obtain

$$a \cong a_0 - \frac{\varepsilon a_0{}^2}{4(3\lambda_1 - \lambda_2)}\left(a_0 - \frac{6b_0}{\lambda_1 + \lambda_2}\right)\left(\frac{e^{2\lambda_1 t} - 1}{2\lambda_1(3\lambda_1 - \lambda_2)} + \frac{2(e^{(\lambda_1 + \lambda_2)t} - 1)}{(3\lambda_1 - \lambda_2)(3\lambda_2 - \lambda_1)} + \frac{e^{2\lambda_2 t} - 1}{2\lambda_2(3\lambda_2 + \lambda_1)}\right)$$
$$b = b_0 - \frac{3\varepsilon a_0^2 b_0}{4}\left(\frac{e^{2\lambda_1 t} - 1}{4\lambda_1^2} + \frac{2(e^{(\lambda_1 + \lambda_2)t} - 1)}{(\lambda_1 + \lambda_2)^2} + \frac{e^{2\lambda_2 t}}{2\lambda_2^2}\right) \tag{19}$$

Hence the first order solution of (10) is

$$x(t,\varepsilon) = \frac{1}{2}a_0(e^{\lambda_1 t} + e^{\lambda_2 t}) + b_0\left(\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}\right) + \varepsilon u_1(a,b,t) \tag{20}$$

where a and b are given by (19) and u_1 is given by (12).

Discussion of Murty's Unified theory: Murty found a unified solution (for un-damped, under damped and over damped case of (1) in the form

$$x(t,\varepsilon) = \rho\cosh\psi + \varepsilon u_1(\rho,\psi) + \ldots \tag{21}$$

or

$$x(t,\varepsilon) = \rho\sinh\psi + \varepsilon u_1(\rho,\psi) + \ldots \tag{22}$$

where ρ and ψ satisfy the first order differential equations

$$\left.\begin{aligned}\dot{\rho} &= -k\rho + \varepsilon A_1(\rho) + ..\\ \dot{\psi} &= \omega_0 + \varepsilon B_1(\rho) + \ldots\end{aligned}\right\} \tag{23}$$

It is interesting to note tnat such type of unified solutions can be found from (5). In this paper, we obtain a solution of (10) in the form of (21). We rewrite (3) as

$$x(t,\,\varepsilon) = a(t)e^{\lambda_1 t} + b(t)e^{\lambda_2 t} + \varepsilon u_1(a,b,t) + \ldots \tag{24}$$

where a and b satisfy the first order differential equations

$$\left.\begin{aligned}\dot{a} &= \varepsilon\tilde{A}_1(a,b,t)) + ..\\ \dot{b} &= \varepsilon\tilde{B}_1(a,b,t)) + ...\end{aligned}\right\} \tag{25}$$

By comparing (3) and (24), we obtain

$$a(t) = \frac{\alpha(t)}{2} + \frac{\beta(t)}{\lambda_1 - \lambda_2},\quad b(t) = \frac{\alpha(t)}{2} - \frac{\beta(t)}{\lambda_1 - \lambda_2} \tag{26}$$

Differentiating (26) with respect to t and utilizing relations of (4) and (25), the following relations between A_1, B_1 and $\tilde{A}_1$, $\tilde{B}_1$ can be found:

$$\tilde{A}_1 = \frac{A_1}{2} + \frac{B_1}{\lambda_1 - \lambda_2},\ \tilde{B}_1 = \frac{A_1}{2} - \frac{B_1}{\lambda_1 - \lambda_2} \tag{27}$$

or,

$$A_1 = \tilde{A}_1 + \tilde{B}_1,\ B_1 = \frac{1}{2}(\lambda_1 - \lambda_2)(\tilde{A}_1 + \tilde{B}) \tag{28}$$

Substituting the values of A_1 and B_1 from (28) into (5) and simplifying, we obtain

$$\left(\frac{\partial\tilde{A}_1}{\partial t} + (\lambda_1 - \lambda_2)\tilde{A}_1\right)e^{\lambda_1 t} + \left(\frac{\partial\tilde{B}_1}{\partial t} + (\lambda_2 - \lambda_1)\tilde{A}_1\right)e^{\lambda_2 t} + \left(\frac{\partial}{\partial t} - \lambda_1\right)\left(\frac{\partial}{\partial t} - \lambda_2\right)u_1 = -f^{(0)} \tag{29}$$

According to then unified theory, the roots of the linear equation of (10) are $\lambda_1 = -k + \omega_0$ and $\lambda_2 = -k - \omega_0$, so that $f^{(00} = e^{-3kt}(a^3e^{3\omega_0 t} + 3a^2be^{\omega_0 t} + 3ab^2e^{-\omega_0 t} + b^3e^{-3\omega_0 t})$. Moreover, in accordance to KBM method, u_1 does not contain terms with $e^{\omega_0 t}$ and $e^{-\omega_0 t}$. Substituting the values of λ_1, λ_2 and $f^{(0)}$ into (29) and assuming that u_1 excludes the terms with $e^{\omega_0 t}$ and $e^{-\omega_0 t}$, we obtain

$$\left(\frac{\partial\tilde{A}_1}{\partial t} + 2\omega_0\tilde{A}_1 = -3a^2be^{-2kt}\right) \tag{30}$$

$$\left(\frac{\partial\tilde{B}_1}{\partial t} + 2\omega_0\tilde{B}_1 = -3ab^2e^{-2kt}\right) \tag{31}$$

and

$$\left(\frac{\partial}{\partial t} + k - \omega_0\right)\left(\frac{\partial}{\partial t} + k + \omega_0\right)u_1 = -e^{-3kt}(a^3e^{3\omega_0 t} + b^3e^{\omega_0 t}) \tag{32}$$

Solving (30)-(32), we obtain

$$\tilde{A}_1 = \frac{3a^2be^{-2kt}}{2(k - \omega_0)},\qquad \tilde{B}_1 = \frac{3ab^2e^{-2kt}}{2(k + \omega_0)} \tag{33}$$

and

$$u_1 = -\frac{e^{-3kt}}{4}\left(\frac{a^3e^{3\omega_0 t}}{(k - \omega_0)(k - 2\omega_0)} + \frac{b^3e^{-3\omega_0 t}}{(k + \omega_0)(k + 2\omega_0)}\right) \tag{34}$$

Substituting the values of $\tilde{A}_1$ and $\tilde{B}_1$ from (33) into (25), we obtain

$$\dot{a} = \frac{3\varepsilon a^2be^{-2kt}}{2(k - \omega_0)},\qquad \dot{b} = \frac{3\varepsilon ab^2e^{-2kt}}{2(k + \omega_0)} \tag{35}$$

Equations of (35) have exact solutions. These equations reduce to

$$\dot{r} = \frac{3\varepsilon kr^3e^{-2kt}}{8\omega^2},\qquad \dot{\phi} = \frac{3\varepsilon\omega_0 r^2e^{-2kt}}{8\omega^2} \tag{36}$$

under the transformations $a = \frac{1}{2}re^{\phi}$, $b = \frac{1}{2}re^{-\phi}$. u_1 in (34) become

$$u_1 = \frac{r^3e^{-3kt}\left((k^2 + 2\omega_0^2)\cosh 3(\omega t + \phi) + 3k\omega\sinh 3(\omega t + \phi)\right)}{16\omega^2(k^2 - 4\omega_0^2)} \tag{37}$$

On the other hand, under the transformations (24) becomes

$$x(t,\varepsilon) = re^{-kt}\cosh(\omega_0 + \phi) + \varepsilon u_1 \tag{38}$$

where u_1 is given by (37), r and ϕ given by (36). Replacing $\rho = re^{-kt}$ and $\psi = \omega_0 t + \phi$, we can show that (38) is a unified solution of (10)[1]. Similarly, we can find the second unified solution of the form (22) from (5).

3. Results and Discussion

In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented KBM method of this article, we refer to the works of Murty *et,.al* [5] (who found an over-damped solution of a second order nonlinear system with constant coefficients), and Shamsul [6-7]. In our present paper, for different initial conditions, w*e* have compared the perturbation solutions (16) of *Duffing's* equations (7) to those obtained by Runge-Kutta Fourth-order procedure.

First of all, x is calculated by (20) with initial conditions $x(0) = 1.00000$ $\dot{x} = 0.00000$ for $\lambda_1 = -1.162$, $\lambda_2 = -.8612$ and $\varepsilon = .1$. The corresponding numerical solution is also computed by Runge-Kutta fourth-order method and is given in the third column of the Table 1. Moreover x is calculated by (38). All the results are shown in Table 1. Percentage errors have also been calculated and given in the fourth column and sixth column of the Table 1. From Table 1, we see that errors for unified solution (38) occur more than 22%, while for the asymptotic solution (20), percentage errors are less than 1.25%. However, when the deference of two roots is much smaller than unity errors occur only 1% (Table 3).

Table 1

t	x_p	x_{nu}	$E^{(1)}\%$	x_u	$E^{(2)}\%$
0.0	1.00000	1.00000	0.00000	1.00000	0.00000
1.0	0.707868	0.716179	-1.1605	0.715523	-0.0916
2.0	0.379374	0.381757	-0.6242	0.338067	-11.4445
3.0	0.183567	0.183943	-0.2044	0.152127	-17.2967
4.0	0.084622	0.084564	0.0686	0.067037	-20.7263
5.0	0.037945	0.037851	0.2483	0.029178	-22.9135

Table 2

t	x_p	x_{nu}	$E^{(1)}\%$
0.0	1.00000	1.00000	0.00000
1.0	0.708669	0.715368	-0.9364
2.0	0.378369	0.379474	-0.2912
3.0	0.181542	0.181205	0.1860
4.0	0.082642	0.082230	0.5010
5.0	0.036457	0.036198	0.7155

4. Conclusion

An asymptotic solution has been obtained for certain over-damped nonlinear systems, which has been found in the sense of extended KBM method, shows a good coincidence with the numerical solution.

References

[1] N.N, Krylov and N.N., Bogoliubov, Introduction to Nonlinear Mechanics. Princeton University Press, New Jersey, 1947.

[2] N. N, Bogoliubov and Yu. Mitropolskii, Asymptotic Methods in the Theory of nonlinear Oscillations, Gordan and Breach, New York, 1961.

[3] Yu.,Mitropolskii, "Problems on Asymptotic Methods of Non-stationary Oscillations" (in Russian), Izdat, Nauka, Moscow, 1964.P. Popov, "A generalization of the Bogoliubov asymptotic method in the theory of nonlinear oscillations", Dokl.Akad. Nauk SSSR 111, 1956, 308-310 (in Russian).

[4] S. N. Murty, B. L. Deekshatulu and G. Krisna, "General asymptotic method of Krylov-Bogoliubov for over-damped nonlinear system", J. Frank Inst. 288 (1969), 49-46.

[5] M.,Shamsul Alam, "A unified Krylov-Bogoliubov-Mitropolskii method for solving nth order nonlinear systems", Journal of the Franklin Institute 339, 239-248, 2002.

[6] M.,Shamsul Alam., "Asymptotic methods for second-order over-damped and critically damped nonlinear system", Soochow J. Math, 27, 187-200, 2001 .

[7] Pinakee Dey, M. Zulfikar Ali, M. Shamsul Alam, An Asymptotic Method for Time Dependent Non-linear Over-damped Systems, J. Bangladesh Academy of sciences., Vol. 31, pp. 103-108, 2007.

[8] Pinakee Dey, Method of Solution to the Over-Damped Non-linear Vibrating System with Slowly Varying Coefficients under Some Conditions, J. Mech. Cont. & Math. Sci. Vol -8 No-1, July, 2013.

[9] H. Nayfeh, Introduction to perturbation Techniques, J. Wiley, New York, 1981.