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Some properties of the line graphs associated to the total graph of a commutative ring

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Abstract: Let R be a commutative ring with identity and $T(\Gamma(R))$ its total graph. The subject of this article is the investigation of the properties of the corresponding line graph $L(T(\Gamma(R)))$. In particular, we determine the girth and clique number of $L(T(\Gamma(R)))$. In addition to that, we find the condition for $L(T(\Gamma(R)))$ to be Eulerian.

Keywords: Total Graph, Line Graph, Commutative Ring

1. Introduction

Let R be a commutative ring with identity, $Z(R)$ be the set of its zero divisors and $Z^*(R) = Z(R) \setminus \{0\}$. There are a number of papers in which the graphs associated to rings were introduced and their properties established. One of the most common is the zero-divisor graph. This idea first appears in [1], where for a ring R, the set of vertices is taken to be R and two vertices x and y are adjacent if and only if $xy = 0$. This work was mostly concerned with colorings of rings. Later, in the paper [2], the authors define the zero-divisor graph $\Gamma(R)$ where the set of vertices is taken to be $Z^*(R)$.

The zero-divisor graph of a commutative ring has also been studied by several other authors. The zero-divisor graph has also been introduced for semigroups and other algebraic structures. There are a lot of interesting questions on graph associated to a given ring. One of the most interesting problems is the question of the embedding of this graph into compact surfaces.

In the paper [3], Anderson and Badawi introduce the total graph $T(\Gamma(R))$ whose set of vertices is R. Two vertices x and y are adjacent if and only if $x + y \in Z(R)$. The question of the embedding of this graph is discussed in [4]. In that paper all isomorphism classes of finite commutative rings whose total graphs are planar or toroidal are listed. The goal of this paper is to study some properties of the corresponding line graph $L(T(\Gamma(R)))$.

Given a graph G, its line graph L(G) is a graph such that every vertex of L(G) represents an edge of G, and two vertices of L(G) are adjacent if and only if their corresponding edges share a common endpoint in G. So, the set of vertices of L(G) is exactly the set of edges of G, and L(G) represents the adjacencies between edges of G. Thus the properties of a graph G that depend only on adjacency between edges may be translated into equivalent properties in L(G) that depend on adjacency between vertices. This is very useful for various problems in graph theory. For example, a matching in G is a set of edges no two of which are adjacent; to a matching in G there corresponds an independent set in L(G), that is a set of vertices in L(G) no two of which are adjacent. If G is connected and if its line graph L(G) is known, one may, according to [5], completely determine G except in the case when L(G) is a triangle. In the paper [6], the authors investigate embeddings of the line graph $L(\Gamma(R))$ and present all isomorphism classes of finite commutative rings such that their line graphs are planar or toroidal.

Knowing the structure of the total graph $L(\Gamma(R))$, our goal is to investigate the structure of its line graph and to look into relations between them. In this paper we determine some properties of this graph. The results related to the embedding problem are given in the paper [7]. There is the list of all isomorphism classes of finite commutative rings for which the associated line graph of the total graph is planar or toroidal. Also, it is shown that for every integer $g \geq 0$ there are only finitely many finite commutative rings such that $\gamma\left(L(T(\Gamma(R))\right) = g$.

For the algebraic part of this paper, notation and terminology is standard and one may find it in [8], or in [9]. For the graph theoretical part, notation and terminology may be find in, e.g., [10], or in [11]. In what follows, all rings R are commutative with identity, $Z(R)$ is the set of zero divisors of $R, Z^*(R) = Z(R) \setminus \{0\}$ and $Reg(R) = R \setminus Z(R)$. By a graph G we mean the simple undirected graph (without loops and parallel edges) with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. The degree of the vertex $v \in V$, denoted $\deg(v)$ is the number of vertices adjacent to the vertex v and

$$\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$$

is the minimal degree of the graph. A graph is regular of the degree r if every vertex has the degree r. The vertices x and y are adjacent if they are connected by an edge. If for every two vertices x and y there exists a path connecting them, then we say that this graph is connected. A graph G is complete if any two vertices are adjacent. If the vertices of the graph G may be separated into two disjoint sets of cardinalities m and n, such that vertices are adjacent if and only if they do not belong to the same set, then the graph G is a complete bipartite graph.

For complete and complete bipartite graphs, we use the notation K^n and $K^{m,n}$, respectively. In particular, $K^{1,n}$ is a star graph. The maximal positive integer r such that $K^r \subseteq G$ for some graph G is the clique number of that graph. For vertices $x, y \in G$ one defines the distance $d(x, y)$, as the length of the shortest path between x and y, if the vertices $x, y \in G$ are connected and $d(x, y) = \infty$, if they are not. Then, the diameter of the graph G is

$$diam(G) = \sup\{\ d(x, y) \mid x, y \in G\ \}.$$

The cycle is a closed path which begins and ends in the same vertex. The cycle of n vertices is denoted by C_n.

The girth of the graph G, denoted by $\mathrm{gr}(G)$ is the length of the shortest cycle in G and $\mathrm{gr}(G) = \infty$, if G has no cycles.

2. On the structure and properties of $L(T\Gamma(R))$

Let R be commutative ring with identity and $T(\Gamma(R))$ is total graph. For simplicity of notation we use $T\Gamma(R)$ for the total graph and $L(T\Gamma(R))$ for its line graph. If for elements $x, y \in R$ one has $x + y \in Z(R)$, then we have a vertex in the graph $L(T\Gamma(R))$ and we denote that vertex by $[x, y]$. From the definition of the graph $T\Gamma(R)$, it follows that the degree of every vertex of this graph depends on number of zero divisors, as well as on whether 2 is zero divisor in R or not.

Proposition 2.1

Let x be a vertex of the graph $T\Gamma(R)$. Then

$$\deg(x) = \begin{cases} |Z(R)| - 1, & 2 \in Z(R) \ \ or \ \ x \in Z(R) \\ |Z(R)|, & otherwise. \end{cases}$$

Proof.

Obviously, if $z_i \in Z(R)$, the vertex $x \in R$ is adjacent to vertices $z_i - x$. Then $\deg(x) = |Z(R)| - 1$ if and only if $x = z_i - x$ for some $z_i \in Z(R)$, i.e., if and only if $2x \in Z(R)$. If $2x \notin Z(R)$, then $\deg(x) = |Z(R)|$.

If $2 \in Z(R)$, then $2x \in Z(R)$ for all $x \in R$, hence $\deg(x) = |Z(R)| - 1$ i.e., all vertices of graph $T\Gamma(R)$ are of degree $|Z(R)| - 1$.

If $2 \notin Z(R)$, then two cases are possible.

Case 1. If $x \in Z(R)$, then $\deg(x) = |Z(R)| - 1$.

Case 2. If $x \notin Z(R)$, then $\deg(x) = |Z(R)|$.

It follows that

$$\deg(x) = \begin{cases} |Z(R)| - 1, & 2 \in Z(R) \ or \ x \in Z(R) \\ |Z(R)|, & otherwise. \end{cases}$$

Theorem 2.2

$$gr(L(T\Gamma(R))) = \begin{cases} 3, & |Z(R)| \geq 3 \quad and \quad R \neq Z_2 \times Z_2 \\ 4, & R \cong Z_2 \times Z_2 \\ \infty, & |Z(R)| \leq 2 \end{cases}$$

Proof.

Case 1. $|Z(R)| = 1$: If R has no proper zero divisors, two cases are possible.

1. If $\mathrm{Char(R)} = 2$, then $L(T\Gamma(R))$ is empty since $T\Gamma(R)$ is totally disconnected.
2. If $\mathrm{Char(R)} \neq 2$, then $T\Gamma(R)$ is the disjoint union of $|R|/2$ graphs K^2 and an isolated vertex 0. In that case, $L(T\Gamma(R))$ is totally disconnected graph with $|R|/2$ vertices. Therefore, $\mathrm{gr}(L(T\Gamma(R))) = \infty$.

Case 2. $|Z(R)| = 2$:

It is known (see [2]) that there are only two non-isomorphic commutative rings containing only one proper zero divisor: Z_4 and $Z_2[x]/(x^2)$. They have isomorphic total graphs - the disjoint union of two complete graphs K^2. It follows that $L(T\Gamma(R))$ is the union of two isolated vertices, therefore $\mathrm{gr}(L(T\Gamma(R))) = \infty$.

Case 3. $|Z(R)| \geq 3$:

3.1. If $Z(R)$ is an ideal and if x, y are different elements from $Z^*(R)$, then $T\Gamma(R)$ contains the triangle $0 - x - y - 0$; so, $L(T\Gamma(R))$ contains the triangle

$$[0, x] - [x, y] - [y, 0] - [0, x].$$

Therefore, $\mathrm{gr}(L(T\Gamma(R))) = 3$.

3.2. If $Z(R)$ is not an ideal, there exist $x, y \in Z^*(R)$ such that $x + y \in Reg(R)$. If one also has $|Z(R)| > 3$, i.e., if there exists $z \in Z^*(R)$ different from x and y, then in $T\Gamma(R)$ there exists a star subgraph $K^{1,3}$ (the vertex 0 is then adjacent to the vertices x, y and z). Since $L(K^{1,3}) = C_3$, then

$$[0, x]-[0, y]-[0, z]-[0, x]$$

is a triangle in $L(T\Gamma(R))$ and $gr(L(T\Gamma(R)) = 3$.

It remains to discuss the case $|Z(R)| = 3$.

According to [2] there are three non-isomorphic commutative rings with three zero divisors: Z_9, $Z_3[x]/(x^2)$ and $Z_2 \times Z_2$. In the first two cases $Z(R)$ is an ideal, while $T\Gamma(Z_2 \times Z_2) = C_4$ Since, $L(C_4) = C_4$ we have that $gr(L(T\Gamma(Z_2 \times Z_2))) = 4$.

In what follows we will denote by $Cl(G)$ the clique number of the graph G.

Theorem 2.3

Let R be a finite commutative ring. Then

$$Cl(L(T\Gamma(R))) = \begin{cases} |Z(R)|-1, & 2 \in Z(R) \\ |Z(R)|, & \text{oherwise}. \end{cases}$$

Proof.

Case 1. Let $2 \in Z(R)$.

According to Proposition 2.1, $\deg(x) = |Z(R)|-1$ for all vertices of the total graph. For some vertex a and the edge ay we have an vertex $[a, y]$ in the line graph. There are $|Z(R)|-1$ such vertices in $L(T\Gamma(R))$. These vertices form subgraph K^r where $r = |Z(R)|-1$. For any set of $l > r$ edges in $T\Gamma(R)$, corresponding vertices in the line graph are not adjacent. Therefore, $Cl(L(T\Gamma(R))) = |Z(R)|-1$.

Case 2. Suppose that $2 \notin Z(R)$.

According to 2.1, $\deg(x) = |Z(R)|$ for some vertex of the total graph. . For some vertex a and the edge ay we have an vertex $[a, y]$ in the line graph. There are $|Z(R)|$ such vertices in $L(T\Gamma(R))$. These vertices form the subgraph K^r where $r = |Z(R)|$. For any set of $l > r$ edges in $T\Gamma(R)$, corresponding vertices in the line graph are not adjacent. So, $Cl(L(T\Gamma(R))) = |Z(R)|$.

Remark. It is known (see [3]) that $T\Gamma(R)$ is disjoint union of the complete graphs K^r ($r = |Z(R)|$), in the case when $Z(R)$ is an ideal and $2 \in Z(R)$, i.e., disjoint union of $K^{r,r}$ complete bipartite graphs in the case when $Z(R)$ is an ideal and $2 \notin Z(R)$. Hence, in the case when $Z(R)$ is an ideal we deal with line graphs of the complete and complete bipartite graphs.

For illustration we give an example of a planar graph and its corresponding line graph which is also planar.

Example 1.

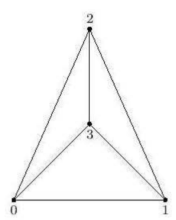

***Figure 1.** Graph* Γ.

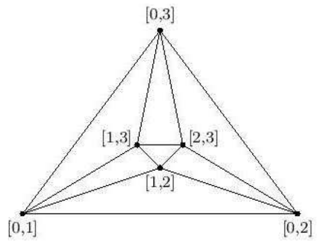

***Figure 2.** Graph* $L(\Gamma)$.

The following is an example of the total graph and subgraph of corresponding line graph which is not planar.

Example 2.

Let $F_4 = Z_2[x]/(x^2+x+1)$. Then we get that $Z(R) = \{v_1, v_2, v_3, v_4, v_5\}$ and $Reg(R) = \{v_6, v_7, v_8,\}$, where $v_1 = (0,0)$, $v_2 = (0,x)$, $v_3 = (0,x+1)$, $v_4 = (0,1)$, $v_5 = (1,0)$, $v_6 = (1,x)$, $v_7 = (1,x+1)$ and $v_8 = (1,1)$. The graph $T\Gamma(R)$ (Figure 3) is regular graph of degree 4 with 8 vertices and 16 edges. Therefore $L(T\Gamma(R))$ is regular graph of degree 6 with 16 vertices and 48 edges (it is clear that the line graph of regular graph of degree r with n vertices is also regular of degree $2r-2$ with $nr/2$ vertices and $nr(r-1)/2$ edges). For simplicity let us denote the vertex $[v_i, v_j]$ of the graph $L(T\Gamma(R))$ by $w_{i,j}$. The line graph $L(T\Gamma(Z_2 \times F_4))$ is not planar because it contains the subgraph Γ_1 (Figure 4) which is a subdivision of K^5.

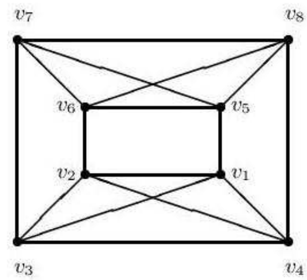

Figure 3. $T\Gamma(Z_2 \times F_4)$.

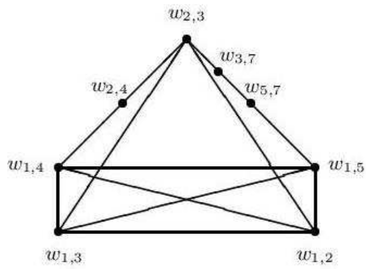

Figure 4. $\Gamma_1 \subseteq L(T\Gamma(Z_2 \times F_4))$.

Remark. Graphs of genus 0 are planar graphs and graphs of genus 1 are toroidal graphs. Euler's Theorem states that if G is a finite connected graph with n vertices, e edges, and of genus g, then

$$n - e + f = 2 - 2g,$$

where f is a number of faces obtained when G is embedded in compact orientable surface which is obtained from sphere by adding g handles. One may think that graph is planar if it can be drawn on the sphere with no edge crossing.

The well-known Kuratowski Theorem states that a graph G is planar if and only if it does not contain a subdivision of K^5 or $K^{3,3}$.

Let R be a finite commutative ring with identity. If $R = Z_2$, then $T\Gamma(R)$ is empty graph. If $R = F_q$, where $q \geq 3$, then $T\Gamma(R)$ is totally disconnected graph with $(q-1)/2$ vertices. In that case, $\deg([u,v]) = 0$ for every vertex $[u,v]$ in $L(T\Gamma(R))$; therefore $\delta(L(T\Gamma(R))) = 0$. In order to exclude trivial cases from discussion, in what follows we assume that R is not a field, i.e. $|Z(R)| \geq 2$.

It is not difficult to see that the following proposition holds.

Proposition 2.4

Let R be a commutative ring such that $T\Gamma(R)$ is connected of finite diameter d. Then the corresponding line graph $L(T\Gamma(R))$ is connected and

$$d-1 \leq d_L \leq d+1$$

where d_L is diameter of $L(T\Gamma(R))$.

Theorem 2.5

Let R be finite commutative ring which is not a field, such that $\mathrm{Char}(R) \neq 2$. Then $L(T\Gamma(R))$ is regular if and only if $2 \in Z(R)$. The degree of regularity is then $2|Z(R)|-4$. In particular, if $|V|$ and $|E|$ denotes the number of vertices and edges of the line graph, then we have that

$$|V| = \frac{|R|(|Z(R)|-1)}{2},$$

$$|E| = \frac{|R|(|Z(R)|-1)(|Z(R)|-2)}{2}.$$

Proof.

Case 1. Suppose first that $2 \in Z(R)$.

Then, according to Proposition 2.1, $T\Gamma(R)$ is regular graph of degree $|Z(R)|-1$ with $|R|(|Z(R)|-1)/2$ edges. Let $[u,v]$ be an arbitrary vertex of the line graph. Then,

$$\deg([u,v]) = \deg(u) + \deg(v) - 2 = 2|Z(R)| - 4,$$

and the claim follows.

Case 2. Suppose that $2 \notin Z(R)$.

Since R is not field, there exists $x \in Z(R) \setminus \{0\}$. Also, holds $1 \neq -1$. So,

$$\deg([0,x]) = 2|Z(R)| - 4 \neq \deg([1,-1]) \\ = 2|Z(R)| - 2.$$

Therefore $L(T\Gamma(R))$ is not regular.

In graph theory, the Eulerian path is a trail in a graph which visits every edge exactly once. Similarly, an Eulerian cycle of the graph G is an Eulerian path which starts and ends on the same vertex. The graph which has an Euler cycle is an Eulerian graph. It is known that the connected undirected graph with at least one edge is Eulerian if and only if all of its vertices have even numbers for degrees.

Thus, we can concentrate on the case when R is a finite ring which is not local. Namely, if R is infinite, it does not make any sense to seek Eulerian cycle and if R is finite and local, then $Z(R) = M$ is an ideal and $T\Gamma(R)$ is not connected. In what follows we will use the fact that any finite commutative ring with identity is isomorphic to a finite direct product of local rings (see [9]). Actually, any Artinian ring is isomorphic to a finite direct product of Artinian local rings. Finite rings forms a special class of Artinian rings. So, we may assume that

$$R \cong R_1 \times R_2 \times \cdots \times R_n,$$

where R_i are finite local rings and $n \geq 2$. It is known that all commutative rings of this form have connected total graph of diameter 2 (see [3]).

Theorem 2.6

Let R be a finite commutative ring which is not local. Then $T\Gamma(R)$ is Eulerian if and only if $2 \in Z(R)$.

Proof.

Let us assume that R is a finite commutative ring and that $2 \in Z(R)$. Then, according to the Proposition 2.1, every vertex of $T\Gamma(R)$ has degree $|Z(R)|-1$ i.e., $T\Gamma(R)$ is regular of degree $|Z(R)|-1$.

Let $[x,y]$ be a vertex of $L(T\Gamma(R))$. Then

$$\deg([x,y]) = \deg(x) + \deg(y) - 2 = 2|Z(R)| - 4$$

is an even number, consequently $L(T\Gamma(R))$ is Eulerian.

Assume now that $2 \notin Z(R)$. Under these conditions, $Z(R)$ is not an ideal, therefore there exist $x, y \in Z(R)$ such that $x + y \notin Z(R)$.

Then the vertices $-x \in Z(R)$ and $x + y \in \mathrm{Reg}(R)$ are adjacent in $T\Gamma(R)$. By Proposition 2.1, $\deg(-x) = |Z(R)|-1$ and $\deg(x+y) = |Z(R)|$. Let us concentrate on the vertex $[-x, x+y]$ in $L(T\Gamma(R))$. We have that

$$\deg([-x, x+y]) = \deg(-x) + \deg(x+y) - 2 \\ = 2|Z(R)| - 3.$$

Thus $L(T\Gamma(R))$ contains a vertex of odd degree, therefore this graph is not Eulerian.

3 Conclusions

In this paper we found some properties of the line graph associated to the total graph of commutative ring. In particular, we determine the girth and clique number of $L(T(\Gamma(R)))$. In addition to that, we find the condition for $L(T(\Gamma(R)))$ to be Eulerian. For further research, it would be interesting to find similar properties for the line graphs associated to the other graphs attached to the commutative rings. For example, one can consider the line graphs of the intersection graphs of ideals of commutative rings.

References

[1] I. Beck, "Coloring of commutative rings", J. Algebra 116, (1988) 208-226.

[2] D.F.Anderson, P.S. Livingston, "The zero-divisor graph of a commutative ring", J. Algebra 217, (1999), 434-447.

[3] D.F. Anderson, A. Badawi, "The total graph of a commutative ring", J. Algebra 2008, 320, 2706-2719.

[4] H.R. Maimani, C. Wickham, S. Yassemi, "Rings whose total graph have genus at most one", Rocky Mountain J. Math., to appear.

[5] H. Whitney, "Congruent graphs and connectivity of graphs", American Journal of Math. 54 (1932), 150-168.

[6] Hung-Jen Chiang-Hsieh, Pei-Feng Lee, Hsin-Ju Wang, "The embedding of line graphs associated to the zero-divisor graphs of commutative rings", Israel J. Math. 180, (2010) 193-222.

[7] Z. Petrović, Z. Pucanović, "The line graph associated to the total graph of a commutative ring", Ars Combinatoria, to appear.

[8] I. Kaplansky, " Commutative rings", Revised Edition, University of Chicago Press, Chicago, 1974.

[9] M.F. Atiyah, I.G. Macdonald, "Introduction to commutative algebra", Adison-Wesley Publishing Company, Reading, MA. 1969.

[10] D.B. West, "Introduction to graph theory", Prantice Hall, Inc., Upper Saddle River, NJ, 1996.

[11] R. Diestel, "Graph Theory", Third Edition, Graduate Texts in Mathematics 173, Springer-Verlag Berlin Heidelberg, 2005.