Pure and Applied Mathematics Journal
2014; 3(6-1): 24-27
Published online December 24, 2014 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.s.2014030601.15
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Catastrophic types depending on degree of non-linearity

Mohammed Nokhas Murad Kaki

Department of Mathematics, Faculty of Science and Science Education, School of Science, Sulaimani University, Sulaimani, Iraq

Email address:
muradkakaee@yahoo.com

To cite this article:
Mohammed Nokhas Murad Kaki. Catastrophic Types Depending on Degree of Non-Linearity. *Pure and Applied Mathematics Journal*. Special Issue: Mathematical Theory and Modeling. Vol. 3, No. 6-1, 2014, pp. 24-27. doi: 10.11648/j.pamj.s.2014030601.15

Abstract: In this paper, we present results on the projection of the folding part of the elementary catastrophe models on the control space to find stability and catastrophic phenomenon of the periodic solutions of some nonlinear differential equations (NLDE) by using methods of catastrophe theory. We have shown here, that the cusp and Butterfly types depend on the degree of nonlinear differential equations, and that the bifurcation can be classified as cusp or butterfly types catastrophe. Moreover, our aim, in this work, is to obtain periodic solutions of some nonlinear differential equation (NLDE) and to study the stability of these periodic solutions and the main result is the following proposition: *The catastrophic Types depending on the degree of non-linear differential equation.*

Keywords: Cusp, Butterfly Catastrophe, Mathematical Model, Stability of Periodic Solution, Bifurcation

1. Introduction

Some characteristics of the phenomena of discontinuous jumping in reality are hard to be explained by equations. The catastrophe theory can explain these characteristics. A cusp and butterfly catastrophe models are developed to analyze the stability by drawing graphs for a cusp and butterfly catastrophe models of nonlinear differential equations, the bifurcation set or the projection of the folding part of the cusp or butterfly catastrophe models on the control space is always accompanied with the saddle-node bifurcation.. The study of catastrophic problems (equilibrium points, catastrophic manifold (CM), amplitude, jump phenomena, …,etc.) has been of immense importance since long time in view of its growing applications in physical, biological and social sciences. Several authors, for example Arrow Smith et al. (1983), Cesari (1971), Hale (1969), Hartman (1963), Hayashi (1964), Hirsch & Smale (1974), Marsden et al. (1976), Sale (1969), Smith et al. (1977), Zeeman (1977, and references therein) and Muhammad Nokhas Murad Kaki (1985) have made their valuable contributions towards studying some aspects (equilibrium points, periodic solutions, limit cycles, stability (or instability), and phenomena associated with forced oscillations) of the problems. To my knowledge, these authors, have, however, not looked into catastrophic problems like saddle- node bifurcation as an catastrophic set, classification and its type and the stability and semi-stability of periodic solutions of NLDE. In the present work, therefore, an effort has been made to study these phenomenon by catastrophic method which might bridge the gap between the above referred works and others in progress both qualitative and quantitative thoughts have been given to the problem so as to present a more clear picture of the physical phenomena. This work may generate a continuous interest to one feel that he has actually available new investigative technique. As well known, there are elementary and non-elementary types of catastrophes; the formers of seven kinds (fold, cusp, swallowtail, hyperbolic, elliptic, butterfly parabolic) and the latter has got no classification, we have shown here that saddle – node bifurcation of the averaged system arising from the general from of the nonlinear differential equation (NLDE) which is of the following form [6]:

$$\ddot{x} + \omega^2 x + \varepsilon f(x.\dot{x}) = 0$$

Some mathematical models that describe a *periodic solution* behavior are considered. Solving of non-linear differential equations and finding stability of solution of them are important work in physical science, and mathematics. Most phenomena in our world are essentially nonlinear and are described by nonlinear ordinary differential equations. Solving nonlinear ordinary differential equation is thus of great importance for gaining insight into real-world or engineering problems. However, generally speaking, it is difficult to obtain accurate solutions of nonlinear problems

[1].So we attempt to solve some nonlinear differential equations by methods of catastrophe theory.

Catastrophe theory is the study of singularity, discontinuity and it is about the possible shapes of the equilibrium states form a catastrophic manifold when we have n state variables in the vector x and k parameters so the possible equilibrium states form this surface, or catastrophic manifold in (n + k)-dimensional space [2]. The behavior of the system over time is described by trajectories over this manifold, so it is important to us to find such a manifold. As well known there are seven types of elementary catastrophe: Fold, Cusp, swallowtail, Hyperbolic umbilici, Elliptic, Butterfly, Parabolic umbilici [2]. We have studded here the butterfly catastrophe and we have related this type with NLDE involves the study of change using the techniques of catastrophe theory[3] spatially we have used the method of Krylov and Bogoliubov to study the stability of periodic solution by using new conditions for the second order nonlinear differential equations:

$$\ddot{x} + \omega^2 x + \varepsilon f(x.\dot{x}) = 0$$

We have shown here that the butterfly catastrophe occurs when the NLDE of fifth degree,. We have divided the main body of this work into three parts; the first part is the introductory in section 2 we have described the method of Krylov and Bogoliupov. in the section 3 we have studded the stability of limit cycles and in section 4 we have studded catastrophic manifold of butterfly.

2. The Method of Krylov and Bogoliubov

Although Krylov and Bogoliubov's method is fairly general, we will apply it only to equations of the form:[6]

$$\ddot{x} + \omega^2 x + \varepsilon f(x.\dot{x}) = 0 \quad (2.1)$$

where ε is small parameter.

For the case ε = 0 we may apply linear theory to obtain the solution:

$$x = A \sin(\omega t + \phi)$$

where A and Ø are arbitrary constants. Differentiating gives:

$$\dot{x} = A\omega\cos(\omega t + \phi)$$

Assume that, for small ε, the solution of (2.1) takes the form:

$$x = A(t)\sin(\omega t + \phi(t)) \quad \dot{x} = A(t)\omega\cos(\omega t + \varphi(t))$$

where A(t) and Ø(t) are slowly varying functions of t.

We proceed to obtain the approximate solution of eq. (1) as follows:

Let

$$y = \dot{x} \quad (2.2)$$

and, from eqs. (2.1) And (2.2), we have

$$\dot{y} = -\omega^2 x - \varepsilon f(x, y) \quad (2.3)$$

To satisfy eqs. (2.2) and (2.3), we further assume that

$$\begin{aligned} x &= A(t)\sin(\omega t + \varphi(t)) \\ \dot{x} &= A(t)\omega\cos(\omega t + \varphi(t)) \end{aligned} \quad (2.4)$$

where $A(t)$ and $\phi(t)$ are slowly varying functions of t, and therefore $\ddot{A}$ and $\ddot{\phi}$ can be neglected. In order that the set of equations (2.4) should be the solutions of equations (2.2) and (2.3) it must satisfy the following conditions[3, 5]

$$\dot{A}\sin\Psi + A\dot{\varphi}\cos\Psi = 0. \quad (2.5)$$

And

$$\dot{A}\omega\cos\Psi - A\omega(\omega + \dot{\varphi})\sin\Psi = -\omega^2 A\sin\Psi - \varepsilon f(x, y). \quad (2.6)$$

Therefore

$$\dot{A}\cos\Psi - A\dot{\varphi}\sin\Psi = -\frac{\varepsilon}{\omega} f(A\sin\Psi, A\omega\cos\Psi) \quad (2.7)$$

Where $\Psi = \omega t + \phi$

Solving (2.5) and (2.7) for $\dot{A}$ and $\dot{\phi}$, we get:

$$\dot{A} = -\frac{\varepsilon}{\omega}\cos\Psi f(A\sin\Psi, A\omega\cos\Psi) \quad (2.8)$$

$$\dot{\varphi} = \frac{\varepsilon}{\omega A}\sin\Psi f(A\sin\Psi, A\omega\cos\Psi) \quad (2.9)$$

Note that A and Ø are both proportional to ε, conferencing that A and Ø are slowly varying functions of time when ε is small and that in terms of the assumption contained in (2.2) and (2.3) equations (2.8)and (2.9) are exact representation of A and Ø..

Krylov and Bogoliubove approximation is to replace A and Ø in equations 2.8 and 2.9 by their average values over one period 2π/ω. A is regarded as a constant in taking the average. This procedure (known as a method of averaging) leads to

$$\dot{A} = -\frac{\varepsilon}{2\pi}\int_0^{2\pi/w}\cos\Psi f(A\sin\Psi, A\omega\cos\Psi)dt \quad (2.10)$$

$$\dot{\varphi} = \frac{\varepsilon}{2\pi A}\int_0^{2\pi/w}\sin\Psi f(A\sin\Psi, A\omega\cos\Psi)dt \quad (2.11)$$

Because $d\Psi = \omega dt$, the substitution $\Psi = \omega t + \phi$ gives the final results

$$\dot{A} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi}\cos\Psi f(A\sin\Psi, A\omega\cos\Psi)d\Psi \quad (2.12)$$

$$\dot{\varphi} = \frac{\varepsilon}{2\pi A\omega}\int_0^{2\pi} \sin\Psi\, f(A\sin\Psi, A\omega\cos\Psi)d\Psi \qquad (2.13)$$

The exact equations 2.8 and 2.9 are thus replaced by approximate equations 2.10 and 2.11. Once the integrals have been evaluated we have first order differential equations to solve for A and Ø. We should find the values of A and Ø by evaluating the integrals 2.12 and 2.13. Then the solution is given approximately by x=A sin (ωt+Ø) whenever A and Ø take their values.

3. Stability of Limit Cycles

A limit cycle is an isolated closed trajectory; this means that its neighboring trajectories are not closed – they spiral either towards or away from the limit cycle. Thus, limit cycles can only occur in nonlinear systems. (In a linear system exhibiting oscillations closed trajectories are neighbored by other closed trajectories. A stable limit cycle is one which attracts all neighboring trajectories. A system with a stable limit cycle can exhibit self-sustained oscillations which are one of the most important phenomena that occur in physical systems. A system oscillates when it has a nontrivial periodic solution. An isolated periodic orbit is called a limit cycle. – most of the Physical and biological processes of interest are of this kind..

A existence of limit cycles:

The amplitudes of possible limit cycles are given by solutions of the equation

$$\dot{A} = 0,\ i.e.\ A\ is\ \text{constant}$$

Now

$$\dot{A} = -\frac{\varepsilon}{2\pi\omega}\int_0^{2\pi} \cos\Psi\ f(A\sin\Psi, A\omega\cos\Psi)d\Psi = G(A),\ say$$

So the amplitudes of the limit cycles are given by the solutions of G(A)=0.

The equation arises whether or not the limit cycle is stable, i.e. if we made a slight disturbance from the limit cycle trajectory in the phase – plane, would the motion return to diverge from the limit cycle.

Consider the expression for A. Suppose that a solution of $G(A) = 0$ is $A = A_1$.

A_1 is the amplitude of a limit cycle, and $G(A_1) = 0$. Now make the disturbance $A = A_1 + \eta$ where η is small. For a stable limit cycle, we require $\eta \to 0$ as $t \to \infty$.

Differentiating we obtain $\dot{A} = \dot{\eta}$.

Also $\dot{A} = G(A_1 + \eta)$
$\approx G(A_1) + \eta G'(A_1)$
$= \eta G'(A_1)$, since $G(A_1) = 0$.

So $\dot{\eta} \approx \eta G'(A_1)$. Solving this equation gives

$\eta \approx Ce^{G(A_1)t}$,where C is a constant. So $\eta \to 0$ as $t \to \infty$ provided $G(A_1) < 0$.

We now have a condition for stability:

1-If $G'(A_1) < 0$, there is a stable limit cycle at $A = A_1$.
2-If $G'(A_1) > 0$, there is an unstable limit cycle at $A = A_1$.

B Non-existence of limit cycles

We turn our attention now to the negative side of the problem of showing limit cycles exist. Here is a theorem which can sometimes be used to show that a limit cycle does not exist.

(Negative Pointcaré - Bendixson Criterion) If, on a simply connected region D of the plane, the expression $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign, then the system, $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ has no periodic orbits lying entirely in *D* of the plane, the expression

4. Catastrophic Manifold of Butterfly

Our purpose, in this section, is to find the catastrophic manifold of butterfly catastrophe, and to show that the butterfly catastrophe occurs in case of NLDE of fifth degree. To do this we first define the function f that represents the butterfly catastrophe. Suppose that the possible equilibrium states of the system are the minima of the function f(x) given by

$$f(x) = x^6 + u_1x^4 + u_2x^3 + u_3x^2 + u_4x \qquad (4.1)$$

The stationary values are given by

$$\partial f/\partial x = 6x^5 + 4u_1x^3 + 3u_2x^2 + 2u_3x + u_4 = 0 \qquad (4.2)$$

The equation (4.2) can have one or three or five real roots.

The second derivative $\partial^2 f/\partial x^2$ is zero on some curve (which includes the point (0, 0, 0, 0, 0) in (x, u_1, u_2, u_3, u_4) – space), and hence that the function has degenerate singularities along the curve.

Also the second derivative can be used to identify the minima; in the case of three real roots, two are minima; and in the case of the single real root, that turns out to be a minimum.

For example take the function f in (2.1) as follows:

$$f(t, u, u) = \mu u^5 + B\sin(wt)$$

The averaged system is:

$$\dot{a} = -\frac{\varepsilon}{w^2}(\beta b + 516\ \mu b r^4 + B$$

$$\dot{b} = \frac{\varepsilon}{w^2}(\beta a + 516\ \mu a r^4)$$

Let μ=615 and the response manifold (RM) is.

$$\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2 = 0$$

Which is a catastrophic manifold of butterfly. And the nonlinear dynamic system is written as follows

$$\dot{\gamma} = -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2) \qquad (4.3)$$

Let $u_1 = 2\beta, u_2 = \beta^2, u_3 = -\beta^2$ and investigate the Liapanov function. Of this dynamic. Construct a function $F(\gamma, u_1, u_2 .u_3) = \frac{1}{6}\gamma^6 + \frac{1}{4}u_1\gamma^4 + u_2\gamma^2 + u_3\gamma$

Which is the Butterfly catastrophe. [5]. It is easily seen that $F(\gamma)$ is a Liapunov function with $\dot{\gamma} = -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2)^2 < 0 \Leftrightarrow -(\gamma^5 + 2\beta\gamma^3 + \beta^2\gamma - B^2) \neq 0$

The degree of F , as well known, is 6 and Butterfly type catastrophe occurs here , that is the type dependents on degree. Now, consider the non-linear differential equation[5] (which is of fifth degree)

$$\ddot{x} + \omega^2 x = \frac{6}{5x^5}$$

The averaged system is

$$\dot{A} = 5/16\ \varepsilon\omega^4 A^5 + 1/2\varepsilon A^3 + \beta^2 A \qquad (4.4)$$

The catastrophic manifold [5] for the averaged system (4.4) is

$$5/16\ \varepsilon\omega^4 A^5 + 1/2\varepsilon A^3 + \beta^2 A = 0, \qquad (4.5)$$

Which represents a butterfly catastrophic model and A represents the amplitude of the solution. . Hence the catastrophic phenomena appear in the system, and, from algebra: The fundamental theorem of algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. This includes polynomials with real coefficients, since every real number is a complex number with zero imaginary part. So, we know that there is at least one root satisfies the eq. (4.5) which represents the amplitude of the periodic solution of the nonlinear differential equation. Now, we study the stability of periodic solution as follows:

Put, $G(A) = 5/16\,\varepsilon\omega^4\, A^5 + 1/2\varepsilon A^3 + \beta^2 A$

If $\frac{\partial}{\partial A} G(A) = {}^{20}\!/_{16}\,\varepsilon\omega^4 A^4 + {}^{3}\!/_{2}\,\varepsilon A^2 + \beta^2 < 0,$

then A is the amplitud of stable periodic solution.

5. Conclusion

The following proposition holds:

The catastrophic Types depending on the degree of nonlinear differential equation.

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