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# New types of chaos and non-wandering points in topological spaces

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**Abstract:** In this paper, we will define a new class of chaotic maps on locally compact Hausdorff spaces called  $\alpha$ -type chaotic maps defined by  $\alpha$ -type transitive maps. This new definition coincides with Devaney's definition for chaos when the topological space happens to be a metric space. Furthermore, we will study new types of non-wandering points called  $\alpha$ -type nonwandering points. We have shown that the  $\alpha$ -type nonwandering points imply nonwandering points but not conversely. Finally, we have defined new concepts of chaotic on topological space. Relationships with some other type of chaotic maps are given.

**Keywords:** Chaos,  $\alpha$ -Type Chaotic Maps,  $\alpha$ -Type Nonwandering Points, Transitive

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## 1. Introduction

I studied a new type of topological transitive map called topological  $\alpha$ -transitive [1] and investigated some of its topological dynamic properties. Further, I introduced and defined the notions of  $\alpha$ -wandering points, and studied the notion of minimal  $\alpha$ -open sets [2]. I have proved some new theorems and propositions associated with these new definitions. I have also shown that topologically  $\alpha$ -type transitive maps,  $\alpha$ -wandering points and topologically  $\alpha$ -mixing are preserved under  $\alpha$ -conjugacy. Recently there has been some interest in the notion of a locally closed subset of a topological space. According to Bourbaki [3] a subset  $S$  of a space  $(X, \tau)$  is called locally closed if it is the intersection of an open set and a closed set. Ganster and Reilly used locally closed sets in [4] to define the concept of LC-continuity, i.e. a function  $f : (X, \tau) \rightarrow (X, \sigma)$  is LC-continuous if the inverse with respect to  $f$  of any open set in  $Y$  is closed in  $X$  [4]. The study of semi open sets and semi continuity in topological spaces was initiated by Levine [5]. Bhattacharya and Lahiri [6] introduced the concept of semi generalized closed sets in topological spaces analogous to generalized closed sets which was introduced by Levine [7]. Throughout this paper, the word "space" will mean topological space. Let  $A$  be a subset of a space  $X$ . Recall that a point  $x$  is said to be an  $\alpha$ -limit point of  $A$  if for each  $\alpha$ -open  $U$  containing  $x$ ,

$U \cap (A \setminus x) \neq \emptyset$ . The set of all  $\alpha$ -limit points of  $A$  is called the  $\alpha$ -derived set of  $A$  and is denoted by  $D_\alpha(A)$ . The point  $x \in X$  is in the  $\alpha$ -closure of a set  $A \subset X$  if  $\alpha(U) \cap A \neq \emptyset$ , for each open set  $U$  containing  $x$ . Then the  $\alpha$ -closure of a set  $A$  is the intersection of all  $\alpha$ -closed sets containing  $A$  and is denoted by  $Cl_\alpha(A)$ . A subset of a space  $X$ , Then the  $int_\alpha(A) = \cup \{U : U \text{ is } \alpha\text{-open and } U \subset A\}$ . In this paper, we will define some new conceptions such as: totally  $\alpha$ -transitive,  $\alpha$ -type hyper-cyclic maps and we proved some theorems associated with this definition. If the map  $f : X \rightarrow X$  is  $\alpha$ -irresolute.  $f$  is said to be weakly topologically  $\alpha$ -mixing if  $f \times f$  is  $\alpha$ -type transitive, i.e. there is a positive integer  $n$  such that  $f^n(U_1) \cap V_1 \neq \emptyset$  and  $f^n(U_2) \cap V_2 \neq \emptyset$  provided that  $U_1, U_2, V_1, V_2$  are non-empty  $\alpha$ -open subsets of  $X$ .  $f$  is said to be topologically  $\alpha$ -mixing if there is a positive integer  $N$  such that  $f^n(U) \cap V \neq \emptyset$  for every  $n > N$  provided  $U$  and  $V$  are non-empty  $\alpha$ -open subsets of  $X$ . Note that a topologically  $\alpha$ -mixing map is weakly topologically  $\alpha$ -mixing and a weakly topologically  $\alpha$ -mixing map is  $\alpha$ -type transitive which is transitive. We will also define a new types of chaotic maps called  $\alpha$ -type chaotic and prove some new theorems associated with this definition. In [10] we defined a new class of chaotic maps on locally compact Hausdorff spaces called  $\lambda$ -type chaotic maps defined by  $\lambda$ -type

transitive maps. This new definition implies John Tylar definition which coincides with Devaney's definition for chaos when the topological space happens to be a metric space. A non-degenerate topological space  $X$  is said to be  $\alpha$ -type chaotic, if for any two distinct points  $p$  and  $q$  of  $X$  there exists an  $\alpha$ -open set  $U$  containing  $p$  and an  $\alpha$ -open set  $V$  containing  $q$  such that no  $\alpha$ -open subset of  $U$  is homeomorphic to any  $\alpha$ -open subset of  $V$ ; and the space  $X$  is said to be strongly  $\alpha$ -type chaotic if for any two distinct points  $p$  and  $q$  of  $X$  there exist  $\alpha$ -open sets  $U$  containing  $p$  and  $V$  containing  $q$  respectively such that no  $\alpha$ -open subset of  $U$  is homeomorphic to any subset of  $V$ . Relationships with some other type of chaotic maps are given

## 2. Preliminaries and Definitions

In this section, we recall some of the basic definitions. Let  $X$  be a topological space and  $A \subset X$ . The interior (resp. closure) of  $A$  is denoted by  $\text{Int}(A)$  (resp.  $\text{Cl}(A)$ ).

**Definition 2.1.** By a topological system we mean a pair  $(X, f)$ , where  $X$  is a compact Hausdorff topological space (the phase space), and  $f: X \rightarrow X$  is a continuous function. The dynamics of the system is given by  $x_{n+1} = f(x_n)$ ,  $x_0 \in X$ ,  $n \in \mathbb{N}$  and the solution passing through  $x$  is the sequence  $\{f(x_n)\}$  where  $n \in \mathbb{N}$ . An element  $x \in X$  is called periodic point if for some  $n \geq 1$ ,  $f^n(x) = x$ . The least such  $n$  is called the period of  $x$ . The set of all periodic points of  $f$  is denoted by  $\text{Per}(f)$ .

**Definition 2.2.** Let  $(X, f)$  be a topological system. Suppose that  $f: X \rightarrow X$  is  $\alpha$ -irresolute map.

1. The map  $f$  is said to be  $\alpha$ -type transitive if there is a positive integer  $n$  such that  $f^n(U) \cap V \neq \emptyset$  provided that  $U$  and  $V$  are non-empty  $\alpha$ -open subsets of  $X$  [1].
2. The map  $f$  is called totally  $\alpha$ -transitive if  $f^n$  is topologically  $\alpha$ -type transitive for all  $n \geq 1$ .

**Definition 2.3.** [1,2] Suppose  $f: X \rightarrow X$  is  $\alpha$ -irresolute map. The map  $f$  is called topologically  $\alpha$ -mixing if, given any nonempty  $\alpha$ -open subsets  $U, V \subseteq X$   $\exists N \geq 1$  such that  $f^n(U) \cap V \neq \emptyset$  for all  $n > N$ . Clearly if  $f$  is topologically  $\alpha$ -mixing then it is also  $\alpha$ -type transitive but not conversely

Let  $(X, f)$  be a topological system. A point  $x \in X$  "moves," its trajectory being the sequence  $x, f(x), f^2(x), \dots, f^n(x), \dots$ , where  $f^n$  is the  $n$ th iteration of the map  $f$ . The point  $f^n(x)$  is the position of  $x$  after  $n$  units of time. The set of points of the trajectory of  $x$  under  $f$  is called the orbit of  $x$ , denoted by  $O_f(x)$ .

**Definition 2.4.** Suppose  $f: X \rightarrow X$  is  $\alpha$ -irresolute map. The map  $f$  is said to be  $\alpha$ -type hyper cyclic if there is a point  $x \in X$  (called  $\alpha$ -type hyper cyclic point) whose orbit under  $f$ ,  $O_f(x) = \{f^n(x) : n \in \mathbb{N}\}$ , is  $\alpha$ -dense in  $X$ .

**Definition 2.5.** Let  $(X, f)$  be a topological system, then the map  $f: X \rightarrow X$  is  $\alpha$ -type chaotic if

1. The set of all periodic points for  $f$  is  $\alpha$ -dense in  $X$ .
2.  $f$  is  $\alpha$ -type hyper cyclic map.

**Proposition 2.6**

1. Let  $X$  be a  $\alpha$ -compact space without isolated point, if  $f$  is  $\alpha$ -type hyper cyclic, that is there exists  $x_0 \in X$  such that the set  $O_f(x_0)$  is  $\alpha$ -dense then  $f$  is  $\alpha$ -type transitive.
2. if  $f: X \rightarrow X$  and  $g: Y \rightarrow Y$  are topologically  $\alpha$ -conjugated by the homeomorphism  $h: Y \rightarrow X$ . Then  $g$  is  $\alpha$ -type hyper cyclic (i.e. for all  $y \in Y$  the orbit  $O_g(y)$  is  $\alpha$ -dense in  $Y$ ) if and only if  $f$  is  $\alpha$ -type hyper cyclic (i.e. the orbit  $O_f(h(y))$  of  $h(y)$  is  $\alpha$ -dense in  $X$ ).
3. If  $X$  is separable and second category then topologically transitive then the map  $f$  is hyper cyclic.

*Proof 1.* Let  $x_0 \in X$  is such that  $O_f(x_0)$  is  $\alpha$ -dense in  $X$ . Given any pair  $U, V$  of  $\alpha$ -open subsets of  $X$ , by  $\alpha$ -density there exists  $n$  such that  $f^n(x_0) \in U$ , but  $O_f(x_0)$  is  $\alpha$ -dense implies that  $O_f(f^n(x_0))$  is  $\alpha$ -dense, so intersects  $V$ , i.e. There exists a positive integer  $m$  such that  $f^m(f^n(x_0)) \in V$ . Therefore  $f^{m+n}(x_0) \in f^m(U) \cap V$  that is  $f^m(U) \cap V \neq \emptyset$ . So  $f$  is  $\alpha$ -type transitive.

*proof 2.* Let  $h: Y \rightarrow X$  be the  $\alpha$ -conjugacy. Assume that  $g$  is  $\alpha$ -type hyper cyclic so there is  $y \in Y$  such that  $O_g(y)$  is  $\alpha$ -dense and let us show that  $f$  is  $\alpha$ -type hyper cyclic i.e.  $O_f(h(y))$  is  $\alpha$ -dense in  $X$ . For any  $U \subset X$  non-empty  $\alpha$ -open set,  $h^{-1}(U)$  is a  $\alpha$ -open set in  $Y$  since  $h^{-1}$  is continuous because  $h$  is  $\alpha$ -homeomorphism and it is non-empty since  $h$  is surjective. By  $\alpha$ -density of  $O_g(y)$ , there exists  $k \in \mathbb{N}$  such that  $g^k(y) \in h^{-1}(U) \Leftrightarrow h^{-1}(g^k(y)) \in U$ . Since  $h$  is a  $\alpha$ -conjugacy,  $f^k \circ h = h \circ g^k$  so  $f^k(h(y)) = h(g^k(y)) \in U$ , therefore  $O_f(h(y))$  intersects  $U$ . This holds for any non-empty  $\alpha$ -open set  $U$  and thus shows that  $O_f(h(y))$  is  $\alpha$ -dense. The other implication follows by exchanging the role of  $f$  and  $g$ .

Suppose  $f: X \rightarrow X$  is  $\alpha$ -irresolute map.  $f$  is said to be weakly topologically  $\alpha$ -mixing if  $f \times f$  is  $\alpha$ -type transitive, i.e. there is a positive integer  $n$  such that  $f^n(U_1) \cap V_1 \neq \emptyset$  and  $f^n(U_2) \cap V_2 \neq \emptyset$  provided that  $U_1, U_2, V_1, V_2$  are non-empty  $\alpha$ -open subsets of  $X$ .  $f$  is said to be topologically  $\alpha$ -mixing if there is a positive integer  $N$  such that  $f^n(U) \cap V \neq \emptyset$  for every  $n > N$  provided  $U$  and  $V$  are non-empty  $\alpha$ -open subsets of  $X$ . It is clear that a topologically  $\alpha$ -mixing map is weakly topologically  $\alpha$ -mixing and a weakly topologically  $\alpha$ -mixing map is  $\alpha$ -type transitive which is transitive.

**Definition 2.7.** A non-degenerate topological space  $X$  is said to be:

- (a)  $\alpha$ -type chaotic if for any two distinct points  $p$  and  $q$  of  $X$  there exists an  $\alpha$ - open set  $U$  containing  $p$  and an  $\alpha$ -open set  $V$  containing  $q$  such that no  $\alpha$ -open subset of  $U$  is homeomorphic to any  $\alpha$ -open subset of  $V$  ;
- (b) strongly  $\alpha$ -type chaotic if for any two distinct points  $p$  and  $q$  of  $X$  there exist  $\alpha$ -open sets  $U$  containing  $p$  and  $V$  containing  $q$  respectively such that no  $\alpha$ -open subset of  $U$  is homeomorphic to any subset of  $V$  ;

Let  $(X, f)$  be a topological system. Suppose  $f : X \rightarrow X$  is  $\alpha$ -irresolute map, the  $\alpha$ - minimality of  $(X, f)$  is defined by requiring that every point  $x \in X$  visit every  $\alpha$ - open set  $V$  in  $X$  (i.e.  $f^n(x) \in V$  for some  $n \in \mathbb{N}$ ) then, instead, one may wish to study the following concept: every nonempty  $\alpha$ -open subset  $U$  of  $X$  visits every nonempty  $\alpha$ -open subset  $V$  of  $X$  in the following sense:  $f^n(U) \cap V \neq \emptyset$ . for some  $n \in \mathbb{N}$ . If the system  $(X, f)$  has this property, then it is called *topologically  $\alpha$ -type transitive* as we mentioned before. We also say that  $f$  itself is topologically  $\alpha$ -type transitive if the system cannot be broken down or decomposed into two  $\alpha$ -subsystems (disjoint sets with nonempty  $\alpha$ -interiors) which do not interact under  $f$ , i.e. , are invariant under the map ( $A \subset X$  is  $f$ -invariant if  $f(A) \subset A$ ).

An  $\alpha$ - minimal topological system is a system that has no non-trivial sub- $\alpha$ -system, that is, any  $\alpha$ -closed set  $A \subset X$  satisfying  $f(A) \subset A$  is either empty or the whole  $X$  itself. Equivalently,  $(X, f)$  is  $\alpha$ -minimal if the orbit of every point  $x$  in  $X$  is  $\alpha$ -dense (i.e.  $Cl_\alpha(O_f(x)) = X$  ). If  $X$  itself is a minimal set we say that the system  $(X, f)$  is a minimal system.

**Definition 2.8.** (Topological weak  $\alpha$ -mixing) A topological system  $(X, f)$  is *topologically weakly  $\alpha$ -mixing* if the product system  $X \times X$  is topologically  $\alpha$ -type transitive. If for every two non-empty  $\alpha$ - open sets  $U, V \subset X$ , all but finitely many time steps  $k \in \mathbb{N}$  satisfy  $f^k(U) \cap V \neq \emptyset$ , then the system is said to be (topologically)  $\alpha$ - mixing. .

In between  $\alpha$ -minimality and topologically  $\alpha$ -type transitivity, we have the notion of strong  $\alpha$ - transitivity.

**Definition 2.9.** A system is strongly  $\alpha$ -transitive if for every point  $x \in X$ . the set  $\bigcup_{n=0}^{\infty} f^{-n}(\{x\})$  is  $\alpha$ -dense, or equivalently, if every non-empty  $\alpha$ -open set  $U \subset X$  satisfies  $\bigcup_{n=0}^{\infty} f^n(U) = X$  .

**Definition 2.10.** (Topologically  $\alpha$ -Exact Map): A map  $f : X \rightarrow X$  is *topologically  $\alpha$ -type exact* if for any non-empty  $\alpha$ -open set  $U \subset X$  there is apposite integer  $n$  for which  $f^n(U) = X$  .

**Proposition 2.11.** We have the following results:

Exact implies mixing implies weakly mixing implies transitive.

Topologically  $\alpha$ -Exact Map implies topologically exact map but not conversely

A non-empty  $\alpha$ - closed invariant set not containing proper

subset which would be  $\alpha$ -closed and invariant is called  $\alpha$ -minimal.

**Theorem 2.12.** Any two  $\alpha$ -minimal sets must have empty intersection.

*Proof.* Let  $M_1$  and  $M_2$  be two distinct  $\alpha$ -minimal sets, and suppose that

$A = M_1 \cap M_2 \neq \emptyset$ . Then  $A$  is  $\alpha$ -closed, and for every  $a \in A$  and every

$n \in \mathbb{N}$ ,  $f^n(a) \in M_1 \cap M_2$ , so  $A$  is invariant. But then  $A$  is a proper subset of both  $M_1$  and  $M_2$  which is  $\alpha$ -closed, invariant and non-empty, contradicting the fact that  $M_1$  and  $M_2$  are  $\alpha$ -minimal.

**Remark 2.13.** It is easy to see that topologically exact maps are also transitive

**Remark 2.14.** It is easy to see that any topologically  $\alpha$ -exact map is also  $\alpha$ -type transitive map which implies transitive map.

**Remark 2.15.** Any topologically  $\alpha$ -exact map implies topologically exact.

**Definition 2.16.** Let  $(X, f)$  be a topological system, then  $f : X \rightarrow X$  is  $\alpha$ -type chaotic if

1. The set of all periodic points for  $f$  is  $\alpha$ -dense in  $X$ .
2.  $f$  is  $\alpha$ -type transitive.

**Theorem 2.17.** Suppose  $f : X \rightarrow Y$  is  $\alpha$ -irresolute map that is onto and suppose that  $D$  is  $\alpha$ -dense subset of  $X$ . Then  $f(D)$  is  $\alpha$ -dense subset of  $Y$  .

**Proposition 2.18.** Recall that if  $(X, f)$  is a topological dynamical system, where  $X$  is a Hausdorff space. Then the following hold:

1. The set of all fixed points is a closed subset of  $X$ .
2. Orbits of any two periodic points are either identical or disjoint.
3. If a trajectory converges, it converges to a fixed point.
4. An element is eventually periodic if and only if it has a finite orbit.
5. Every orbit is an invariant set; the orbits of periodic points are minimal invariant sets.
6. A subset of  $X$  is invariant if and only if it is a union of orbits.
7. The closure of an invariant set is also invariant.
8. The set of all periodic points is an invariant set.
9. For each  $A \subset X$ , the set  $\bigcup_{n=0}^{\infty} f^n(A)$  is the smallest

invariant set containing  $A$ .

Let  $(X; f)$  and  $(Y; g)$  be two topological systems. Then a topological conjugacy  $h$  from  $f$  to  $g$  carries orbit of  $f$  passing through  $x$  to orbit of  $g$  passing through  $h(x)$ .

**Theorem 2.20.** Let  $(X; f)$  and  $(Y; g)$  be two topological systems and let  $h : X \rightarrow Y$  be a topological  $\alpha$ -conjugacy[1]. Then

1.  $h^{-1} : Y \rightarrow X$  is a topological  $\alpha$ -conjugacy.
2.  $h \circ f^n = g^n \circ h$  for all  $n \in \mathbb{N}$ .
3.  $p \in X$  is a periodic point of  $f$  if and only if  $h(p)$  is a periodic point of  $g$ .

1. If  $p$  is a periodic point of  $f$  with  $\alpha$ -neighborhood  $D$  of  $p$ , then the  $\alpha$ -neighborhood of  $h(p)$  is  $h(D)$ .
2. The periodic points of  $f$  are  $\alpha$ -dense in  $X$  if and only if the periodic points of  $g$  are  $\alpha$ -dense in  $Y$ .
3.  $f$  is topologically  $\alpha$ -type transitive on  $X$  if and only if  $g$  is topologically  $\alpha$ -type transitive on  $Y$ .
4.  $f$  is topologically  $\alpha$ -minimal map on  $X$  if and only if  $g$  is topologically  $\alpha$ -minimal map on  $Y$ .
5.  $f$  is topologically  $\alpha$ -mixing map on  $X$  if and only if  $g$  is topologically  $\alpha$ -mixing map on  $Y$ .
6.  $f$  is  $\alpha$ -type chaotic map on  $X$  if and only if  $g$  is  $\alpha$ -type chaotic map on  $Y$ .

Let  $(X, f)$  be a topological system. A map  $f: X \rightarrow X$  is called  $\alpha$ -type chaotic, if it is topological  $\alpha$ -type transitive and, its periodic points are  $\alpha$ -dense in  $X$ , i.e. every non-empty  $\alpha$ -open subset of  $X$  contains a periodic point

*Definition 2.21.* Let  $(X, f)$  be a topological system. A point  $x \in X$  is called  $\alpha$ -recurrent if for every  $\alpha$ -open set  $V$  containing  $x$ , there is  $n \in \mathbb{N}$  such that  $f^n(x) \in V$ .

*Proposition 2.22.* Every  $\alpha$ -recurrent point is recurrent point but not conversely.

*Theorem 2.23.* [1] Let  $(X, \tau)$  be a topological space and  $f: X \rightarrow X$  be  $\alpha$ -irresolute map. Then the following statements are equivalent:

- (1)  $f$  is topological  $\alpha$ -transitive map
- (2) For every nonempty  $\alpha$ -open set  $U$  in  $X$ ,  $\bigcup_{n=0}^{\infty} f^n(U)$  is  $\alpha$ -dense in  $X$
- (3) For every nonempty  $\alpha$ -open set  $U$  in  $X$ ,  $\bigcup_{n=0}^{\infty} f^{-n}(U)$  is  $\alpha$ -dense in  $X$
- (4) If  $B \subset X$  is  $\alpha$ -closed and  $B$  is  $f$ -invariant i.e.  $f(B) \subset B$ . then  $B=X$  or  $B$  is nowhere  $\alpha$ -dense
- (5) If  $U$  is  $\alpha$ -open and  $f^{-1}(U) \subset U$  then  $U=\emptyset$  or  $U$  is  $\alpha$ -dense in  $X$ .

For proof see [1]

### 3. The Product of Two Topological Systems

Now, given two maps  $f: X \rightarrow X$  and  $g: Y \rightarrow Y$  on topological spaces  $X$  and  $Y$ . respectively, consider their product  $f \times g: X \times Y \rightarrow X \times Y$ ,  $(f \times g)(x, y) = (f(x), g(y))$ , with product topology on  $X \times Y$ .

*Lemma 3.1.* Let  $(X, f), (Y, g)$  be topological systems. The set of periodic points of  $f \times g$  is  $\alpha$ -dense in  $X \times Y$  if and only if, for both of  $f$  and  $g$ , the sets of periodic points in  $X$  and  $Y$  are  $\alpha$ -dense in  $X$ , respectively  $Y$ .

Proof: Assume that the set of periodic points of  $f$  is  $\alpha$ -dense in  $X$  (i.e.  $Cl_\alpha(Per(f)) = X$ ) and the set of periodic points of  $g$  is  $\alpha$ -dense in  $Y$  (i.e.  $Cl_\alpha(Per(g)) = Y$ ). We have to prove that the set of periodic points of  $f \times g$  is  $\alpha$ -dense in  $X \times Y$ . Let  $W \subset X \times Y$  be any non-empty

$\alpha$ -open set. Then there exist non-empty  $\alpha$ -open sets  $U \subset X$  and  $V \subset Y$  with  $U \times V \subset W$ . By assumption, there exists a point  $x \in U$  such that  $f^n(x) = x$  with  $n \geq 1$ . Similarly, there exists  $y \in V$  such that  $g^m(y) = y$  with  $m \geq 1$ . For  $p = (x, y) \in W$  and  $k = mn$  we get

$$\begin{aligned} (f \times g)^k(p) &= (f \times g)^k(x, y) \\ &= ((f^k(x), g^k(y))) = (x, y) = p \end{aligned}$$

Therefore  $W$  contains a periodic point and thus the set of periodic points of  $f \times g$  is  $\alpha$ -dense in  $X \times Y$ .

Conversely let  $U \subset X$  and  $V \subset Y$  be non-empty  $\alpha$ -open subsets. Then  $U \times V$  is a non-empty  $\alpha$ -open subset of  $X \times Y$ . As the set of the periodic points of  $f \times g$  is  $\alpha$ -dense in  $X \times Y$ , there exists a point  $p = (x, y) \in U \times V$  such that  $(f \times g)^n(x, y) = ((f^n(x), g^n(y))) = (x, y)$  for some  $n$ . From the last equality we obtain  $f^n(x) = x$  for  $x \in U$  and  $g^n(y) = y$  for  $y \in V$ .

By Lemma 3.1,  $\alpha$ -denseness of periodic points carry over from factors to products. But, topological  $\alpha$ -type transitivity may not carry over to products. The converse of this situation is however true:

*Lemma 3.2.* Let  $f: X \rightarrow X$  and  $g: Y \rightarrow Y$  be maps and assume that the product  $f \times g$  is topological  $\alpha$ -type transitive on  $X \times Y$ . Then the maps  $f$  and  $g$  are both topological  $\alpha$ -type transitive on  $X$  and  $Y$  respectively.

Proof. We prove the  $\alpha$ -transitivity of  $f$ ; the  $\alpha$ -transitivity of  $g$  can be proved similarly. Let  $U_1, V_1$  be non-empty  $\alpha$ -open sets in  $X$ . Then the sets  $U = U_1 \times Y$  and  $V = V_1 \times Y$  are  $\alpha$ -open in  $X \times Y$ . As  $f \times g$  is  $\alpha$ -type transitive, there exists a positive integer  $n$  such that  $(f \times g)^n(U) \cap V \neq \emptyset$ . From the equalities:

$$\begin{aligned} (f \times g)^n(U) \cap V &= [f^n(U_1) \times g^n(Y)] \cap [V_1 \times Y] \\ &= [f^n(U_1) \cap V_1] \times [g^n(Y) \cap Y] \neq \emptyset. \end{aligned}$$

So  $f^n(U_1) \cap V_1 \neq \emptyset$ . Thus  $f$  is topological  $\alpha$ -type transitive.

*Definition 3.3.* Let  $f: X \rightarrow X$  be a map on the topological space  $X$ . If for every nonempty  $\alpha$ -open subsets  $U, V \subset X$  there exists a positive integer  $n_0$  such that for every  $n \geq n_0$ ,  $f^n(U) \cap V \neq \emptyset$  then  $f$  is called topologically  $\alpha$ -type mixing.

It is clear that topological  $\alpha$ -type mixing implies topological  $\alpha$ -type transitive.

There is an even stronger notion that implies topological  $\alpha$ -type mixing.

*Definition 3.4.* Let  $f: X \rightarrow X$  be a map on the topological space  $X$ . If for every nonempty  $\alpha$ -open subset  $U \subset X$  there is a positive integer  $n_0$  such that for every  $n \geq n_0$ ,  $f^n(U) = X$ , then  $f$  is called locally  $\alpha$ -type

eventually onto.

*Lemma 3.5.* The product of two topologically  $\alpha$ -type mixing maps is topologically  $\alpha$ -type mixing.

*Proof:* Let  $(X, f)$ ,  $(Y, g)$  be topological systems and  $f, g$  be topologically  $\alpha$ -type mixing maps. Given  $W_1, W_2 \subset X \times Y$ , there exists  $\alpha$ -open sets  $U_1, U_2 \subset X$  and  $V_1, V_2 \subset Y$ , such that  $U_1 \times V_1 \subset W_1$  and  $U_2 \times V_2 \subset W_2$ . By assumption there exist  $n_1$  and  $n_2$  such that

$$f^k(U_1) \cap U_2 \neq \emptyset \text{ for } n \geq n_1 \text{ and } g^k(V_1) \cap V_2 \neq \emptyset \text{ for } n \geq n_2.$$

$$\text{For } n \geq n_0 = \max\{n_1, n_2\}$$

we get

$$[(f \times g)^k(U_1 \times V_1)] \cap (U_2 \times V_2) = [f^k(U_1) \times g^k(V_1)] \cap (U_2 \times V_2)$$

$$= [f^k(U_1) \cap U_2] \times [g^k(V_1) \cap V_2] \neq \emptyset$$

Which means that  $f \times g$  is topologically  $\alpha$ -type mixing.

Now, we give some sufficient conditions for a product map to be  $\alpha$ -type chaotic.

*Theorem 3.6.* Let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be  $\alpha$ -type chaotic and topologically  $\alpha$ -type mixing maps on topological spaces  $X$  and  $Y$ . Then  $f \times g : X \times Y \rightarrow X \times Y$  is  $\alpha$ -type chaotic.

*Proof:* The map  $f \times g$  has  $\alpha$ -dense periodic points by Lemma 3.1 and it is topologically  $\alpha$ -type mixing by Lemma 3.5 and hence topologically  $\alpha$ -type transitive. Thus the two conditions of  $\alpha$ -type chaos are satisfied.

## 4. $\alpha$ -Minimal Maps and $\alpha$ -Non-Wandering Points

In the study of the dynamics of  $\alpha$ -irresolute map  $f : X \rightarrow X$  of  $\alpha$ -compact space  $X$  into itself, a central role is played by the various low recursive properties of the points of  $X$ . One of the important such properties is  $\alpha$ -non-wanderingness. A point  $x \in X$  is called  $\alpha$ -wandering if there exists  $\alpha$ -open set  $U$  containing  $x$  such that for all  $n > 0$ ,  $f^n(U) \cap U = \emptyset$ . A point is  $\alpha$ -non-wandering if it is not a  $\alpha$ -wandering point. The  $\alpha$ -non-wandering set is the complement of the set of  $\alpha$ -wandering points. We will prove that the  $\alpha$ -wandering set is  $\alpha$ -open and the  $\alpha$ -nonwandering set is  $\alpha$ -closed. It is easy to show that the  $\alpha$ -non-wandering set  $\alpha\Omega(f)$  is a non-empty  $\alpha$ -closed invariant subset of  $X$ . A non-wandering set of a topological system has the property that an orbit starting at any point of the set comes arbitrarily close arbitrarily often to the set. Examples of  $\alpha$ -non-wandering sets are fixed points, limit cycles, invariant sets.

One of the goals of dynamical system theory is to decompose the  $\alpha$ -nonwandering set in to disjoint  $\alpha$ -closed subsets, called  $\alpha$ -type basic sets, which have  $\alpha$ -dense orbits, when this can be done, the entire phase space  $X$  can be partitioned into the  $\alpha$ -stable sets of the  $\alpha$ -type basic sets. The  $\alpha$ -stable set of a  $\alpha$ -type basic set is the set of points whose  $w$ -limit is in the  $\alpha$ -type basic set. But, the  $\alpha$ -unstable set of  $\alpha$ -type basic set is the set of points with  $\alpha$ -limit set in the

$\alpha$ -type basic set. If  $X$  is a compact space then every limit set is nonempty.

*Proposition 4.1.* If  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  are topologically  $\alpha$ -conjugate. Then

- (1)  $f$  is  $\alpha$ -type transitive if and only if  $g$  is  $\alpha$ -type transitive;
- (2)  $f$  is  $\alpha$ -minimal if and only if  $g$  is  $\alpha$ -minimal;
- (3)  $f$  is topologically  $\alpha$ -mixing if and only if  $g$  is topologically  $\alpha$ -mixing.

*Definition 4.2.* Let  $f : X \rightarrow X$  be  $\alpha$ -irresolute self-map of a topological space  $X$ . A fundamental  $\alpha$ -type domain for  $f$  is  $\alpha$ -open subset  $D \subset X$  such that every orbit of  $f$  intersect  $D$  in at most one point and intersect  $Cl_\alpha(D)$  in at least one point.

*Proposition 4.3.* Let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be two  $\alpha$ -irresolute self-maps. Assume that there are a fundamental  $\alpha$ -type domain  $D_f \subset X$  for  $f$ , a fundamental  $\alpha$ -type domain  $D_g \subset Y$  for  $g$  and a  $\alpha$ -homeomorphism  $h : Cl_\alpha(D_f) \rightarrow Cl_\alpha(D_g)$  such that  $g \circ h = h \circ f$  on  $f^{-1}(Cl_\alpha(D_f)) \cap Cl_\alpha(D_f)$ . Then  $f$  and  $g$  are topologically  $\alpha$ -conjugated.

*Definition 4.4.* Let  $(X; f)$  be a topological system. A point  $x \in X$  is  $\alpha$ -type non-wandering if for any  $\alpha$ -open set  $U$  containing  $x$  there is  $N > 0$  such that  $f^N(U) \cap U \neq \emptyset$ . The set of all  $\alpha$ -type non-wandering point is denoted by  $NW_\alpha(f)$  or  $\alpha\Omega(f)$ . A point which is not  $\alpha$ -type non-wandering is called  $\alpha$ -type wandering or  $\alpha$ -wandering for short.

*Proposition 4.5.* Let  $(X, f)$  be a topological system every  $\alpha$ -type non-wandering point in  $X$  is non-wandering point, but not conversely.

*Theorem 4.6.* Let  $(X, f)$  be a topological system on  $\alpha$ -Hausdorff space  $X$ . Then:

- (1)  $NW_\alpha(f)$  is  $\alpha$ -closed.
- (2)  $NW_\alpha(f)$  is  $f$ -invariant.
- (3) If  $f$  is invertible, then  $NW_\alpha(f^{-1}) = NW_\alpha(f)$ .
- (4) If  $X$  is  $\alpha$ -compact then  $NW_\alpha(f) \neq \emptyset$ .
- (5) If  $x$  is  $\alpha$ -type non-wandering point in  $X$ , then for every  $\alpha$ -open set  $U$  containing  $x$  and  $n_0 \in \mathbb{N}$  there is  $n > n_0$  such that  $f^n(U) \cap U \neq \emptyset$ .

## 5. Conclusion

There are the main results:

*Definition 5.1.* Suppose  $f : X \rightarrow X$  is  $\alpha$ -irresolute map. The map  $f$  is said to be  $\alpha$ -type hyper cyclic if there is a point  $x \in X$  (called  $\alpha$ -type hyper cyclic point) whose orbit under  $f$ ,  $O_f(x) = \{f^n(x) : n \in \mathbb{N}\}$ , is  $\alpha$ -dense in  $X$ .

*Definition 5.2.* Let  $(X, f)$  be a topological system, then the map  $f : X \rightarrow X$  is  $\alpha$ -type chaotic if

1. The set of all periodic points for  $f$  is  $\alpha$ -dense in  $X$ .

2.  $f$  is  $\alpha$ -type hyper cyclic map.

*Theorem 5.3.* Suppose  $f : X \rightarrow Y$  is  $\alpha$ -irresolute map that is onto and suppose that  $D$  is  $\alpha$ -dense subset of  $X$ . Then  $f(D)$  is  $\alpha$ -dense subset of  $Y$ .

*Definition 5.4.* Let  $(X, f)$  be a topological system. A point  $x \in X$  is called  $\alpha$ -recurrent if for every  $\alpha$ -open set  $V$  containing  $x$ , there is  $n \in \mathbb{N}$  such that  $f^n(x) \in V$ .

*Proposition 5.5.* Every  $\alpha$ -recurrent point is recurrent point but not conversely.

*Theorem 5.6.* Any two  $\alpha$ -minimal sets must have empty intersection.

*Lemma 5.7.* Let  $(X, f)$ ,  $(Y, g)$  be topological systems. The set of periodic points of  $f \times g$  is  $\alpha$ -dense in  $X \times Y$  if and only if, for both of  $f$  and  $g$ , the sets of periodic points in  $X$  and  $Y$  are  $\alpha$ -dense in  $X$ , respectively  $Y$ .

*Lemma 5.8.* Let  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  be maps and assume that the product  $f \times g$  is topological  $\alpha$ -type transitive on  $X \times Y$ . Then the maps  $f$  and  $g$  are both topological  $\alpha$ -type transitive on  $X$  and  $Y$  respectively.

*Lemma 5.9.* The product of two topologically  $\alpha$ -type mixing maps is topologically  $\alpha$ -type mixing.

*Definition 5.10.* Let  $(X, f)$  be a topological system. A point  $x \in X$  is  $\alpha$ -type non-wandering if for any  $\alpha$ -open set  $U$  containing  $x$  there is  $N > 0$  such that  $f^N(U) \cap U \neq \emptyset$ . The set of all  $\alpha$ -type non-wandering point is denoted by  $NW_\alpha(f)$  or  $\alpha\Omega(f)$ . A point which is not  $\alpha$ -type non-wandering is called  $\alpha$ -type wandering or  $\alpha$ -wandering for short.

*Proposition 5.11.* Let  $(X, f)$  be a topological system every  $\alpha$ -type non-wandering point in  $X$  is non-wandering point, but not conversely.

*Proposition 5.12* Let  $(X, f)$  be a topological system on  $\alpha$ -Hausdorff space  $X$ . Then:

- (1)  $NW_\alpha(f)$  is  $\alpha$ -closed.
- (2)  $NW_\alpha(f)$  is  $f$ -invariant.
- (3) If  $f$  is invertible, then  $NW_\alpha(f^{-1}) = NW_\alpha(f)$ .
- (4) If  $X$  is  $\alpha$ -compact then  $NW_\alpha(f) \neq \emptyset$ .
- (5) If  $x$  is  $\alpha$ -type non-wandering point in  $X$ , then for every  $\alpha$ -open set  $U$  containing  $x$  and  $n_0 \in \mathbb{N}$  there is  $n > n_0$  such that  $f^n(U) \cap U \neq \emptyset$ .

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