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Some Separate Quasi-Asymptotics Properties of Multidimensional Distributions and Application

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Abstract: Quasi-asymptotic behavior of functions as a method has its application in observing many physical phenomena which are expressed by differential equations. The aim of the asymptotic method is to allow one to present the solution of a problem depending on the large (or small) parameter. One application of asymptotic methods in describing physical phenomena is the quasi-asymptotic approximation. The aim of this paper is to look at the quasi-asymptotic properties of multidimensional distributions by extracted variable. Distribution $T(x_0,x)$ from $S'(\bar{R}_+^1\times R^n)$ has the property of the separability of variables, if it can be represented in form $T(x_0,x)=\sum\varphi_i(x_0)\psi_i(x)$ where distributions, $\varphi_i(x_0)$ from $S'(\bar{R}^1)$ and ψ_i from $S(R^n)$, x_0 from $\bar{R}^1_+$ and x is element R^n different values of do not depend on each other. Distribution $T(x_0,x)$ the element $S'(\bar{R}_+^1\times R^n)$ is homogeneous and of order α at variable x_0 is element $\bar{R}^1_+$ and $x=x_1,x_2,\dots,x_n$ from R^n if for k>0 it applies that $T(kx_0,kx)=k^\alpha T(x_0,x)$. The method of separating variables is one of the most widespread methods for solving linear differential equations in mathematical physics. In this paper, the results by V. S Vladimirov are used to present the proof of the basic theorems, regarding the quasi-asymptotic behavior of multidimensional distributions by a singular variable, with the application of quasi-asymptotics to the solution of differential equations.

Keywords: Distribution Spaces, Asymptotics, Separate Quasi-Asymptotics, Multidimensional Distributions

1. Introduction

We use $S(\mathbb{R}_0^n)$ to mark the standard space of the Schwartz's rapidly decreasing functions, and $S'(\mathbb{R}_0^n)$ to mark the corresponding space of the slowly increasing distributions [1, 7].

If $f(t)\in S'$ and $\varrho(k)$ is a positive and a continuous function for $k>0$, distribution $f(t)$ has quasi-asymptotics at infinity (at zero) with respect to positive function $\varrho(k)$, if the following is valid

$$\frac{1}{\varrho(k)}f(kt)\to g(t),\left(\frac{1}{\varrho(k)}f\left(\frac{t}{k}\right)\to g(t)\right) \qquad (1)$$

$k\to\infty$ in $S'(\mathbb{R}_0^n)$ with the distribution being $g(t)S'(\mathbb{R}_0^n)$, [1, 2-4, 7].

If $g(t)=0$ distribution $f(t)$ has a trivial quasi-asymptotics at infinity (that is, at zero) with respect to positive function $\varrho(k)$. If (1) is true, function $\varrho(k)$ occurs as an auto-modal function. If $\varrho(k)$ is a positive and continuous function, and $k\to\infty$, then we say that $\varrho(k)$ is an auto-modal function, if for a real number $a>0$ there exists

$$\lim_{k\to\infty}\frac{\rho(ak)}{\rho(k)}=a^\alpha \qquad (2)$$

Where by it converges evenly along a on each compact semi-axies $(0,\infty)$. Distribution $f(t)\in S'_+$ is asymptotically homogeneous with respect to function $\varrho(k)$ of order α if:

$$\frac{1}{\varrho(k)}f(kt)\to C\cdot f_{\alpha+1}(t)\text{ in }S_+' \qquad (3)$$

where the nucleus of fractional differentiation and integration $f_\alpha(t)\in S'$ is defined by

$$f_\alpha(t)=\begin{cases}\frac{\theta(t)\cdot t^{\alpha-1}}{\Gamma(\alpha)}, if & \alpha>0\\ \frac{d^N}{dt^N}f_{\alpha+N}(t), if & \alpha\le 0,\alpha+N>0\end{cases} \qquad (4)$$

with $\Gamma(\alpha)$ being the gamma function, and $\theta(t)$ being the Heaviside function [1-3, 6, 7].

The fractional derivative, [3-5, 7], of order α and distribution of $f(t) \in \mathcal{S}'(\mathbb{R}_1)$ is defined by the formula

$$f^{(-\alpha)}(t) = f_\alpha(t) * f(t) \quad (5)$$

Distribution $T(x_0, x) \in \mathcal{S}'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ has the property of the separability of variables, if it can be represented in form $T(x_0, x) = \sum_i \varphi_i(x_0)\psi(x)$ where distributions $\varphi(x_0) \in \mathcal{S}'(\mathbb{R}_+^1)$ and $\psi \in \mathcal{S}(\mathbb{R}^n$, $x_0 \in \mathbb{R}_+^1$ and $x \in \mathbb{R}^n$ for different values of i do not depend on each other, [15].

Distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ is homogeneous and of order α at variable$x_0 \in \overline{\mathbb{R}_+^1}$and $x = x_1, x_2, \dots, x_n \in \mathbb{R}^n$ if for $k > 0$ it applies that $T(kx_0, kx) = k^\alpha T(x_0, x)$, [1, 3, 7, 8].

In other words, distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1 \times \mathbb{R}^n})$ is homogeneous and of order α at variable x_0 and x if for each test function $\phi(x_0, x) \in \mathcal{S}(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ the following is valid

$$\langle T(x_0, x), \phi\left(\frac{x_0}{k}, \frac{x}{k}\right)\rangle = k^{2\alpha+n+1}\langle T(x_0, x), \phi(x_0, x)\rangle,$$

$k > 0$. Indeed,

$$\langle T(kx_0, kx), \phi(x_0, x)\rangle =$$

$$= \begin{Bmatrix} \text{shift } kx_0 = x_0' \Rightarrow x_0 = \dfrac{x_0'}{k}, \\ kx = x' \Rightarrow x = \dfrac{x'}{k} \end{Bmatrix}$$

$$= \frac{1}{k^{n+1}}\langle T(x'_0, x'), \phi\left(\frac{x'_0}{k}, \frac{x'}{k}\right)\rangle = \begin{pmatrix} x'_0 = x_0 \\ x' = x \end{pmatrix}$$

$$= \frac{1}{k^{n+1}}\langle T(x_0, x), \phi\left(\frac{x_0}{k}, \frac{x}{k}\right)\rangle \quad (6)$$

For example, let it be that $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ in the form of $T(x_0, x) = f(x_0) \times g(x)$ with distributions $f(x_0) \in \mathcal{S}'({\mathbb{R}_1}^+), g(x)\mathcal{S}'(\mathbb{R}_0^n)$ being homogenous and of order α. Then, there is a number of equations that are valid:

$$\langle T(kx_0, kx), \phi(x_0, x)\rangle = \langle f(kx_0) \times g(kx), \phi(x_0, x)\rangle$$

$$= \langle f(kx_0), \langle g(kx), \phi(x_0, x)\rangle\rangle = \begin{pmatrix} f(kx_0) = k^\alpha f(x_0) \\ g(kx) = k^\alpha g(x) \end{pmatrix}$$

$$= k^{2\alpha}\langle f(x_0), \langle g(x), \phi(x_0, x)\rangle\rangle$$

$$= k^{2\alpha}\langle T(x_0, x), \phi(x_0, x)\rangle. \quad (7)$$

From (6) and (7) we can see that the following equation is valid

$$\frac{1}{k^{n+1}}\langle T(x_0, x), \phi\left(\frac{x_0}{k}, \frac{x}{k}\right)\rangle = k^{2\alpha}\langle T(x_0, x), \phi(x_0, x)\rangle$$

and from here, there is

$$\langle T(x_0, x), \phi\left(\frac{x_0}{k}, \frac{x}{k}\right)\rangle = k^{2\alpha+n+1}\langle T(x_0, x), \phi(x_0, x)\rangle.$$

For example, distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ in the form of $T(x_0, x) = f(x_0) \times g(x)$ with $f(x_0) \in \mathcal{S}'({\mathbb{R}_1}^+), g(x)\mathcal{S}'(\mathbb{R}_0^n)$, and $f(x_0)$ being homogenous and of order α, and test function $\phi(x_0, x)$ in the fom of $\phi(x_0, x) = \sum_i {\varphi_i}^1(x_0)\varphi_i^2(x)$ with $\varphi_i^1(x_0) \in S({\mathbb{R}_1}^+), \varphi_i^2(x) \in S(\mathbb{R}^n)$ and $\varphi^1(x_0)$, is a homogenous distribution of order α and then, for $(\forall i)$ the following is true

$$\langle T(kx_0, x), \phi(x_0, x)\rangle = \langle f(kx_0) \times g(x), \phi(x_0, x)\rangle$$

$$= \langle f(kx_0), \langle g(x), \phi(x_0, x)\rangle\rangle$$

$$= \langle f(kx_0). \langle g(x), \textstyle\sum_i {\varphi_i}^1(x_0)\varphi_i^2(x)\rangle\rangle$$

$$= \sum_i \langle f(kx_0), {\varphi_i}^1(x_0)\rangle\ \langle g(x), \varphi_i^2(x)\rangle$$

$$= k^\alpha \sum_i \langle f(kx_0), {\varphi_i}^1(x_0)\rangle\ \langle g(x), \varphi_i^2(x)\rangle$$

$$= k^\alpha\langle f(x_0)g(x), \phi(x_0, x)\rangle = k^\alpha\langle T(x_0, x), \phi(x_0, x)\rangle$$

since the set of functions $\sum_i {\varphi_i}^1(x_0)\varphi_i^2(x)$ is dense in $\mathcal{S}(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$. This is followed by the claim, because it is valid in a dense set, and with its continuity, it extends to entire set in $\mathcal{S}(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$. Homogeneity by the second variable is similarly defined [1, 6].

The homogeneity of distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ separable at variable x_0 assuming that distribution $\varphi_i^1(x_0) \in S({\mathbb{R}_1}^+)$ is homogeneous and of order α for each *i*, then, form these relations, it follows that

$$T(kx_0, x) = \sum_i \varphi_i(kx_0)\psi(x)$$

$$= k^\alpha \textstyle\sum_i \varphi_i(x_0)\psi(x) = k^\alpha T(x_0, x). \quad (8)$$

Homogeneity at variable x is similarly observed.

Let there be distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$. Distribution $T(x_0, x)$ with $x_0 \in \mathbb{R}_+^1$ and $x \in \mathbb{R}^n$ has quasi-asymptotic at infinity at variable x_0 relative to auto-modal function ϱ, if there is distribution $G(x_0, x) \neq 0$ such that

$$\lim_{k\to\infty} \frac{1}{\rho(k)} T(kx_0, x) = G(x_0, x) \text{ in } \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n). \quad (9)$$

Quasi-asymptotics by the separated variable at zero is similarly defined, [1, 7].

Let us suppose a distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$. Distribution $T(x_0, x)$, $x_0 \in \mathbb{R}_+^1$ and $x \in \mathbb{R}^n$ has quasi-asymptotics at zero at variable x_0 with respect to auto-modal function ϱ, if, and only if there is distribution $G(x_0, x) \neq 0$ such that

$$\lim_{k\to\infty} \frac{1}{\rho(k)} T\left(\frac{x_0}{k}, x\right) = G\left(\frac{x_0}{k}, x\right) \neq 0 \quad (10)$$

in $\mathcal{S}'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$.

For distributions from $D'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$ (or $\mathcal{S}'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$ we define the fractional (rational) differentiation at variable x_0 as a convolution $f_\alpha(x_0)$ with $f(x_0, x)$ at, x_0 by the following formula

$$f^{(\alpha)} = f_{-\alpha}(x_0) \ast f(x_0, x) \quad (11)$$

which belongs to $D'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$ if $f \in D'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$ that is, $\mathcal{S}'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$ if $f \in \mathcal{S}'\left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right)$, (more in [3-6, 8, 12, 13]).

2. Some Quasi-Asymptotics Properties of Multidimensional Distributions

We provide proof of some of the basic theorems that apply to multidimensional distributions, and their formulaic presentation can be seen in [1].

Theorem 1. *If distribution* $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ *is asymptotically homogeneous with respect to positive function* $\rho(k)$ *at variable* x_0 *or if the following is true*

$$\lim_{k\to\infty} \frac{1}{\rho(k)} T(kx_0, x) = G(x_0, x) \text{ in } \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n) \quad (12)$$

then $\rho(k)$ *is an auto-modal function*.

Proof: Let (12) be true and let $\phi(x_0, x) \in S(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ test function such that $\langle G(x_0, x), \phi(x_0, x) \rangle \neq 0$.

Then let the test function be of the following form $\phi(x_0, x) = \sum_i \varphi_i{}^1(x_0)\varphi_i^2(x)$, so that $\varphi_i^1(x_0) \in S(\mathbb{R}_1{}^+), \varphi_i^2(x) \in S(\mathbb{R}^n)\ \forall i$, are continuous functions with the following feature:

$$supp\ \varphi_i^1 \subset \overline{\mathbb{R}_+^1}, supp\ \varphi_i^2 \subset \mathbb{R}_+^n,$$

$$supp\ \phi = supp\ \varphi_i^1 \times supp\ \varphi_i^2 \subset \left(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n\right), (\forall i),$$

$$K \subset \mathbb{R}_+^1 \text{ compact set.}$$

For $\phi(x_0, x)$ and $a \in K$ it applies that

$$\frac{1}{a}\phi\left(\frac{x_0}{a}, x\right) = \frac{1}{a}\sum_i \varphi_i^1\left(\frac{x_0}{a}\right)\varphi_i^2(x).$$

Now, the following is valid for distribution $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ and test function

$$\phi(x_0, x) \in S(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n);$$

$$\langle T(kx_0, x), \frac{1}{a}\phi\left(\frac{x_0}{a}, x\right) \rangle \xrightarrow{k\to\infty, a\in K} \langle G(x_0, x), \frac{1}{a}\phi\left(\frac{x_0}{a}, x\right) \rangle$$

For $a \in K$, and using $(x_0 = ax_0')$ the following is valid

$$\frac{\rho(ak)}{\rho(k)}\langle\frac{T(kx_0, x)}{\rho(ak)}, \frac{1}{a}\phi\left(\frac{x_0}{a}, x\right)\rangle$$

$$= \frac{\rho(ak)}{\rho(k)}\langle\frac{T(akx'_0, x)}{\rho(ak)}, \varphi(x'_0, x)\rangle$$

$$= \frac{\rho(ak)}{\rho(k)}\langle\frac{T(akx_0, x)}{\rho(ak)}, \phi(x_0, x)\rangle$$

$$\xrightarrow{k\to\infty, a\in K} \langle G(x_0, x), \frac{1}{a}\phi\left(\frac{x_0}{a}, x\right) \rangle \quad (13)$$

Further, if we replace k with ak, $a \in K$, the following is valid

$$\frac{1}{\rho(ak)}\langle T(akx_0, x), \phi(x_0, x)\rangle \xrightarrow{k\to\infty} \langle G(x_0, x), \phi(x_0, x)\rangle. \quad (14)$$

Using relations (12) and (13), we get the following relation

$$\frac{\rho(ak)}{\rho(k)} \xrightarrow{k\to\infty} \frac{\langle G(x_0, x), \frac{1}{a}\phi\left(\frac{x_0}{a}, x\right) \rangle}{\langle G(x_0, x), \phi(x_0, x)\rangle}. \quad (15)$$

From here, by inserting $\left(ax_0' = x_0\right)$ we get the following

$$\frac{\rho(ak)}{\rho(k)} \xrightarrow{k\to\infty} \frac{\langle G(ax'_0, x), \phi(x'_0, x) \rangle}{\langle G(x_0, x), \phi(x_0, x)\rangle}.$$

From here, we get the required relation

$$\frac{\rho(ak)}{\rho(k)} \xrightarrow{k\to\infty} \frac{\langle G(ax_0, x), \phi(x_0, x) \rangle}{\langle G(x_0, x), \phi(x_0, x)\rangle} = C(a).$$

From the existence of $\lim_{k\to\infty} \frac{\rho(ak)}{\rho(k)} = C(a)$ following $C(a) = a^\alpha$ and $\rho(a) = a^\alpha L(a)$, and Karamata L function [16], it follows that function $\rho(k)$ is an auto-modal function, even in the case of multi-variable distributions.

Theorem 2. *Let distribution* $T(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ *be asymptotically homogeneous with respect to positive function* $\rho(k)$ *at variable* x_0. *In this case, if the order of auto-modal function* $\rho(k)$ *is equal to* α *, then distribution* $G(x_0, x)$ *in the following equation*

$\lim_{k\to\infty} \frac{1}{\rho(k)} T(kx_0, x) = G(x_0, x)$ *is equal to*

$G(x_0, x) = Cf_\alpha(x_0) \times g(x)$, *with* C *being the constant.*

Proof: It has already been shown in the case of distributions of one variable [1],[7], that distribution $G(x) \in S_+{}'$ has the form of $G(x) = Cf_\alpha(x)$ with C being the constant, and $f_\alpha(x)$ being the nucleus of fractional differentiation.

For $G(x_0, x) \in \mathcal{S}'(\overline{\mathbb{R}_+^1} \times \mathbb{R}^n)$ let us suppose that $f(x_0) \in \mathcal{S}'(\overline{\mathbb{R}_+^1})$, and $g(x) = S'(\mathbb{R}^n)$, and that distribution $f(x_0)$ is homogeneous and of order α, and $G(x_0, x) = f(x_0) \times g(x)$. Since for the function in the form of $\varphi(x_0)\psi(x) \in \mathcal{S}'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ the following applies

$$\lim_{k\to\infty} \langle\langle\frac{1}{\rho(k)} T(x_0, x), \varphi(x_0)\rangle, \psi(x) \rangle$$

$$=\langle Cg(x)f_{\alpha+1}(x_0), \psi(x)\rangle = Cf_{\alpha+1}(x_0)\langle g(x), \psi(x)\rangle$$

so distribution $g(x) = S'(\mathbb{R}^n)$, $G(x_0, x)$ is in the form of $G(x_0, x) = Cf_{\alpha+1}(x_0) \times g(x)$.

Theorem 3. *If distribution* $T(x_0, x)$ *is separated at variable forms* x_0 *, then it has the following form:*

$T(x_0, x) = T_1 g_1(x) + T_2(x_0)g_2(x_0)$ *and distribution* $T(x_0, x)$ *has the quasi-asymptotics of order* α *in relation to function* $k^\alpha\rho(k)$ *at a variable* x_0, *if* T_1 *and* T_2 *have the same quasi-asimptotics in relation to function* $\rho(k)$. *The reverse of the theorem is not valid.*

Proof. Let us show that distribution $T(x_0, x)$ is has quasi-asymptotics of order α with respect to $\rho(k)$ if T_1 and T_2 have the same quasi- asymptotics. Let the test function $\phi(x_0, x)$ be in the form of $\phi(x_0, x) = \sum_i \varphi_i(x_0)\psi_i(x)$. By the definition of quasi-asymptotics, the following applies:

$$\frac{1}{\rho(k)}\langle T(kx_0,x),\phi(x_0,x)\rangle = \langle\frac{T(kx_0,x)}{\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\langle\frac{T_1(kx_0)g_1(x)+T_2(kx_0)g_2(x)}{k^\alpha\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\langle\frac{T_1(kx_0)g_1(x)}{\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle+\langle\frac{T_2(kx_0)g_2(x)}{\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle.$$

Since $\rho(k)=k^\alpha L(k)$ and $T_1(kx_0)=k^\alpha\, T_1(x_0)$ and $T_2(kx_0)=k^\alpha\, T_2(x_0)$ therefore

$$=\langle\frac{T_1(kx_0)g_1(x)}{k^\alpha L(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle+\langle\frac{T_2(kx_0)g_2(x)}{k^\alpha L(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\frac{1}{k^\alpha L(k)}\langle T_1(kx_0),\langle g_1(x),\sum_i \varphi_i(x_0)\psi_i(x)\rangle\rangle+\frac{1}{k^\alpha L(k)}\langle T_2(kx_0),\langle g_2(x),\sum_i \varphi_i(x_0)\psi_i(x)\rangle\rangle$$

$$=\frac{1}{k^\alpha L(k)}\sum_i\langle T_1(kx_0),\varphi_i(x_0)\rangle\langle g_1(x),\psi_i(x)\rangle+\frac{1}{k^\alpha L(k)}\sum_i\langle T_2(kx_0),\varphi_i(x_0)\rangle\langle g_2(x),\psi_i(x)\rangle$$

$$=\frac{k^\alpha}{k^\alpha L(k)}\sum_i\langle T_1(kx_0),\varphi_i(x_0)\rangle\langle g_1(x),\psi_i(x)\rangle+\frac{k^\alpha}{k^\alpha L(k)}\sum_i\langle T_2(kx_0),\varphi_i(x_0)\rangle\langle g_2(x),\psi_i(x)\rangle$$

$$=\frac{1}{L(k)}\sum_i\langle T_1(kx_0),\varphi_i(x_0)\rangle\langle g_1(x),\psi_i(x)\rangle+\frac{1}{L(k)}\sum_i\langle T_2(kx_0),\varphi_i(x_0)\rangle\langle g_2(x),\psi_i(x)\rangle$$

$$=\frac{1}{L(k)}\langle T_1(x_0)g_1(x),\sum_i \varphi_i(x_0)\psi_i(x)\rangle+\frac{1}{L(k)}\langle T_2(x_0)g_2(x),\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\frac{1}{L(k)}\langle T_1(x_0)g_1(x),\phi(x_0,x)\rangle+\frac{1}{L(k)}\langle T_2(x_0)g_2(x),\phi(x_0,x)\rangle$$

$$=\frac{1}{L(k)}\langle T_1(x_0)g_1(x)+T_2(x_0)g_2(x),\phi(x_0,x)\rangle.$$

This shows that distribution $T(x_0,x)=T_1g_1(x)+T_2(x_0)g_2(x_0)$ has quasi-asymptotics of order α with respect to function $k^\alpha\rho(k)$ at variable x_0 if distributions T_1 and T_2 have the same quasi-asymptotics.

The reverse of the theorem is not valid. To show this, it is enough to show that, for example, the following is not valid for distribution $T_1(x_0)=x_0^{\alpha+1}+x_0^\alpha$ and $T_2(x_0)=-x_0^{\alpha+1}+x_0^\alpha$ with respect to function $k^\alpha\rho(k)$. Indeed

$$\frac{1}{\rho(k)}\langle T(kx_0,x),\phi(x_0,x)\rangle$$

$$=\langle\frac{T(kx_0,x)}{\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle = \langle\frac{k^\alpha x_0^\alpha[kx_0(g_1(x)-g_2(x))+g_1(x)+g_2(x)]}{\rho(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\langle\frac{k^\alpha x_0^\alpha\left[kx_0\left(g_1(x)-g_2(x)\right)+g_1(x)+g_2(x)\right]}{k^\alpha L(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\langle\frac{x_0^\alpha[kx_0(g_1(x)-g_2(x))+g_1(x)+g_2(x)]}{L(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle$$

$$=\langle\frac{x_0^\alpha[(kx_0+1)g_1(x)+(kx_0-1)g_2(x)]}{L(k)},\sum_i \varphi_i(x_0)\psi_i(x)\rangle.$$

From here, it can be seen that $T(x_0,x)$ has no quasi-asymptotics when $k\to\infty$.

Theorem 4. *In order for distribution* $T(x_0,x)\in\mathcal{S}'(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n)$ *to be asymptotically homogeneous at infinity, with respect to auto-modal function* $\rho(k)$ *at variable* x_0, *it is necessary, and it is also sufficient, that for each* $\beta\in\mathbb{R}$ *its* *fractional derivative $T^{(-\beta)}(x_0,x)$ is asymptotically homogeneous with respect to $k^{\beta}\rho(k)$.*

Proof: We define fractional differentiation in $\mathcal{S}'(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n)$ with distribution $\underline{T(x_0,x)}$ at x_0 as convolution of distribution $f_\beta(x_0)\in\mathcal{S}'(\overline{\mathbb{R}_+^1})$ and distribution $T(x_0,x)\in\mathcal{S}'(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n$ i.e. $T^{(-\beta)}(x_0,x)=T(x_0,x)*f_\beta(x_0)$. Using the property of distribution $f_\beta(x_0)$ to be homogeneous and of order $\beta-1$, that is, using the validity of the following $f_\beta(kx_0)=k^{\beta-1}f_\beta(x_0)$, we get the following:

$\lim_{k\to\infty}\frac{1}{k^{\beta}\rho(k)}\langle T^{(-\beta)}(kx_0,x),\phi(x_0,x)\rangle$, from here, if we put that , $kx_0=x'_0$, we get

$$=\lim_{k\to\infty}\frac{1}{k^{\beta+1}\rho(k)}\langle T^{(-\beta)}(x'_0,x),\phi\left(\frac{x'_0}{k},x\right)\rangle$$

$$=\lim_{k\to\infty}\frac{1}{k^{\beta+1}\rho(k)}\langle T^{(-\beta)}(x_0,x),\phi\left(\frac{x_0}{k},x\right)\rangle.$$

By using the definition of convolution

$$T(x_0,x)*f_\beta(x_0)=\frac{1}{\Gamma(\beta)}\Theta(x_0)x_0^{\beta-1}*T(x_0,x)$$

$$=\frac{1}{\Gamma(\beta)}\int_0^\infty (x,x_0)^{\beta-1}\,T(t,x)\,dt=T^{(-\beta)}(x_0,x).$$

we can see that the last equation is precisely the β primitive integral for $T(x_0,x)$. Based on this, we have that $T(x_0,x)\in\mathcal{S}'(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n, f_\beta(x_0)\in\mathcal{S}'(\overline{\mathbb{R}_+^1})$,

$$\langle T(x_0,x)*f_\beta(x_0),\phi(x_0,x)\rangle$$

$$=\lim_{k\to\infty}\langle T(x_0,x)*f_\beta(\tau),\eta_k(x_0,\tau)\phi(x_0+\tau,x)\rangle,$$

with $\{\eta_k\}$ being unit sequence. If there is a limes on the right-hand side for each series $\{\eta_k,k\to\infty\}$ then the function from $S(\mathbb{R}^2)$ which converges to number one in $\mathbb{R}^2$ and this limit does not depend on the choice of series $\{\eta_k,k\to\infty\}$ then we have that $T(x_0,x)*f_\beta(x_0)\in S'(\mathbb{R}^{n+1})$. Based on this, the last equation transforms into

$$\lim_{k\to\infty}\frac{1}{k^{\beta+1}\rho(k)}\langle T(x_0,x)*f_\beta(x_0),\phi\left(\frac{x_0}{k},x\right)\rangle$$

$$=\lim_{k\to\infty}\frac{1}{k^{\beta+1}\rho(k)}\langle T(x_0,x)\times\ f_\beta(\tau),\eta_k(x_0;\tau)\phi\left(\frac{x_0+\tau}{k},x\right)\rangle.$$

Now, if we put that

$$\left(\begin{array}{c}\tau=k\tau'\\ \dfrac{x_0+\tau}{k}=\dfrac{x_0+k\tau}{k}=\dfrac{x_0}{k}+\tau'\end{array}\right)$$

the last equation transforms into the following form:

$$\lim_{k\to\infty}\frac{1}{k^{\beta}\rho(k)}\langle T(x_0,x),\langle f_\beta(k\tau'),\phi\left(\frac{x_0}{k}+\tau',x\right)\rangle\rangle$$

$$\lim_{k\to\infty}\frac{1}{k^{\beta}\rho(k)}\langle T(x_0,x),\langle f_\beta(k\tau),\phi\left(\frac{x_0}{k}+\tau,x\right)\rangle\rangle$$

(since $f_\beta(k\tau)=k^{\beta-1}f_\beta(\tau)$)

$$\lim_{k\to\infty}\frac{k^{\beta-1}}{k^{\beta}\rho(k)}\langle T(x_0,x),\langle f_\beta(\tau),\phi\left(\frac{x_0}{k}+\tau,x\right)\rangle\rangle.$$

From the last equation, using the shift ($x_0=kx'_0$ we get the following

$$\lim_{k\to\infty}\frac{1}{\rho(k)}\langle T(kx'_0,x),\langle f_\beta(\tau),\phi(x'_0+\tau,x)\rangle\rangle$$

$$\lim_{k\to\infty}\frac{1}{\rho(k)}\langle T(kx_0,x),\langle f_\beta(\tau),\phi(x_0+\tau,x)\rangle\rangle$$

$\lim_{k\to\infty}\frac{1}{\rho(k)}\langle T(kx_0,x),\psi(x_0,x)\rangle$,where function

$\phi(x_0,x)=\psi(x_0,x)=\langle f_\beta(\tau),\phi(x_0+\tau,x)\rangle$ creates the auto-morphism of space $S(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n)\to S(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n)$.

Theorem 5. *Let it be that $m\in\mathbb{N}_0$ and that $T(x_0,x)\in\mathcal{S}'(\overline{\mathbb{R}_+^1}\times\mathbb{R}^n)$ has quasi-asymptotics $g(x_0,x)$ at variable x_0 with respect to auto-modal function $\rho(k)$, $k\to\infty$ and let it be that ${x_0}^m\in\mathcal{M}_{(,)}$, with $\mathcal{M}_{(,)}$ being the space of the multiplier of distributions, then distribution ${x_0}^m\cdot T(x_0,x)$ also has quasi-asymptotics $G(x_0,x)={x_0}^m\cdot g(x_0,x)$ at x_0 with respect to auto-modal function $k^m\rho(k)$.*

Proof. There is

$$\lim_{k\to\infty}\langle\frac{(kx_0)^m\cdot T(kx_0,x)}{k^m\cdot\rho(k)},\phi(x_0,x)\rangle=$$

$$=\lim_{k\to\infty}\langle\frac{k^m\cdot x_0^m\cdot T(kx_0,x)}{k^{|m|}\cdot\rho(k)},\phi(x_0,x)\rangle$$

$$=\lim_{k\to\infty}\langle\frac{T(kx_0,x)}{\rho(k)},x_0^m\phi(x_0,x)\rangle$$

$$=\lim_{k\to\infty}\langle\frac{T(kx_0,x)}{\rho(k)},x_0^m\phi(x_0,x)\rangle=\langle g(x_0,x),x_0^m\phi(x_0,x)\rangle$$

$$=\langle x_0^m g(x_0,x),\phi(x_0,x)\rangle=\langle G(x_0,x),\phi(x_0,x)\rangle.$$

From here we find that $G(x_0,x)={x_0}^m\cdot g(x_0,x)$.

3. Example of the Use of Quasi-Asymptitics to the Solutions of Differential Equations

Let L be a differential operator with constant coefficients $a_\beta(x)=a_\beta$ and let $f\in\mathcal{D}'$, be such a distribution that convolution $\mathcal{E}*f$ exists in $\mathcal{D}'$ where $\mathcal{E}\in\mathcal{D}'$ is the fundamental solution of equation $L(D)\mathcal{E}=\delta(x)$, [3, 6, 9, 11].

Then the solution $u=E*f$ of differential equation $L(D)u=f(x), f\in\mathcal{D}'$ has quasi-asymptotics of order α with respect to $\rho(k)=k^\alpha L(k)$ (with $L(k)$ being the Karamata slow-varying function), if distribution $f\in\mathcal{D}'$ has such quasi-asymptotics, $\mathcal{D}'$-distribution space.

Proof: Let f have the quasi-asymptotics with respect to $\rho(k)=k^\alpha L(k)$. Then the following is valid

$$\frac{1}{\rho(k)}\langle f(kx),\phi(x)\rangle=\frac{1}{k\rho(k)}\langle f(x),\phi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\ \delta(x)*f(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\ L(D)\mathcal{E}*f(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\left(\sum_{|\alpha|=0}^{m}a_\alpha D^\alpha\mathcal{E}(x)\right)*f(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\sum_{|\alpha|=0}^{m}a_\alpha D^\alpha\ (\mathcal{E}*f)(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\ L(D)(\mathcal{E}*f)(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\ L(D)u(x),\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\sum_{|\alpha|=0}^{m}a_\alpha D^\alpha u(x)\ ,\varphi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\sum_{|\alpha|}^{m}\langle D^\alpha u(x),a_\alpha\phi\left(\frac{x}{k}\right)\ \rangle$$

$$=\frac{1}{k\rho(k)}\sum_{|\alpha|}^{m}(-1)^{|\alpha|}\ \langle u(x),D^\alpha\left(a_\alpha\phi\left(\frac{x}{k}\right)\right)\ \rangle$$

$$=\frac{1}{k\rho(k)}\langle\ u(x),L^*(D)\ \phi\left(\frac{x}{k}\right)\rangle$$

$$=\frac{1}{k\rho(k)}\langle\ u(kx),L(-D)\ \phi(x)\rangle\ =\langle\ \frac{u(kx)}{\rho(k)},L(-D)\ \phi(x)\rangle.$$

Therefore, we have the following: $\frac{1}{\rho(k)}\langle f(kx),\phi(x)\rangle=\langle\ \frac{u(kx)}{\rho(k)},L(-D)\ \phi(x)\rangle$, and, as per assumption, $f$has the quasi-asymptotics, thus, distribution u has one also.

4. Conclusion

Most of the theorems proved in this paper on quasi-asymptotics of distributions at a separable variable have their analog in the case of one-dimensional distributions. In [1], Vladimirov showed a theorem that does not have a one-dimensional analog, the consequence of which is very important, and on the basis of which the application of separated quasi-asymptotics in to the solutions of differential equations.

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