Some Separate Quasi-Asymptotics Properties of Multidimensional Distributions and Application

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Abstract: Quasi-asymptotic behavior of functions as a method has its application in observing many physical phenomena which are expressed by differential equations. The aim of the asymptotic method is to allow one to present the solution of a problem depending on the large (or small) parameter. One application of asymptotic methods in describing physical phenomena is the quasi-asymptotic approximation. The aim of this paper is to look at the quasi-asymptotic properties of multidimensional distributions by extracted variable. Distribution $T(x_0, x)$ from $S'(\mathbb{R}^{n+1}) \times \mathbb{R}^n$ has the property of the separability of variables, if it can be represented in form $T(x_0, x) = \sum \phi_i(x_0) \psi_i(x)$ where distributions, $\phi_i(x_0)$ from $S'(\mathbb{R}^{1+n})$ and $\psi_i$ from $S(\mathbb{R}^n)$, $x_0$ from $\mathbb{R}^1$ and $x$ is element $\mathbb{R}^n$ different values of do not depend on each other. Distribution $T(x_0, x)$ the element $S'(\mathbb{R}^{1+n}) \times \mathbb{R}^n$ is homogeneous and of order $\alpha$ at variable $x_0$ element $\mathbb{R}^1$ and $x=x_1, x_2, \ldots, x_n$ from $\mathbb{R}^n$ if for $k>0$ it applies that $T(kx_0, kx) = k^{\alpha} T(x_0, x)$. The method of separating variables is one of the most widespread methods for solving linear differential equations in mathematical physics. In this paper, the results by V. S Vladimirov are used to present the proof of the basic theorems, regarding the quasi-asymptotic behavior of multidimensional distributions by a singular variable, with the application of quasi-asymptotics to the solution of differential equations.

Keywords: Distribution Spaces, Asymptotics, Separate Quasi-Asymptotics, Multidimensional Distributions

1. Introduction

We use $S(\mathbb{R}^n)$ to mark the standard space of the Schwartz’s rapidly decreasing functions, and $S'(\mathbb{R}^n)$ to mark the corresponding space of the slowly increasing distributions [1, 7].

If $f(t) \in S'$ and $g(k)$ is a positive and a continuous function for $k > 0$, distribution $f(t)$ has quasi-asymptotics at infinity (at zero) with respect to positive function $g(k)$, if the following is valid

$$1 \over g(k) f(kt) \rightarrow g(t), \left(1 \over g(k) \right) f \left( \frac{k}{x} \right) \rightarrow g(t) \quad (1)$$

$k \rightarrow \infty$ in $S'(\mathbb{R}^n)$ with the distribution being $g(t) S'(\mathbb{R}^n),$ [1, 2-4, 7].

If $g(t) = 0$ distribution $f(t)$ has a trivial quasi-asymptotics at infinity (that is, at zero) with respect to positive function $g(k)$. If (1) is true, function $g(k)$ occurs as an auto-modal function. If $g(k)$ is a positive and continuous function, and $k \rightarrow \infty$, then we say that $g(k)$ is an auto-modal function, if for a real number $\alpha > 0$ there exists

$$\lim_{k \rightarrow \infty} \frac{g(ak)}{g(k)} = a^\alpha \quad (2)$$

Where by it converges evenly along $a$ on each compact semi-axies ($0, \infty$). Distribution $f(t) \in S'_+ \mathbb{R}^n$ is asymptotically homogeneous with respect to function $g(k)$ of order $\alpha$ if:

$$1 \over g(k) f(kt) \rightarrow C \cdot f_{a+1}(t) \quad (3)$$

where the nucleus of fractional differentiation and integration $f_a(t) \in S'$ is defined by

$$f_a(t) = \left\{ \begin{array}{ll} \theta(t) t^{\alpha-1} & {\text{if}} \quad \alpha > 0 \\ \frac{d}{dt} f_{a+N}(t) & {\text{if}} \quad \alpha \leq 0, \alpha + N > 0 \\ \end{array} \right. \quad (4)$$

with $\Gamma(\alpha)$ being the gamma function, and $\theta(t)$ being the Heaviside function [1-3, 6, 7].

The fractional derivative, [3-5, 7], of order $\alpha$ and
distribution of \( f(t) \in S'(\mathbb{R}_1) \) is defined by the formula
\[
f^{(-\alpha)}(t) = f_\alpha(t) * f(t)
\]  
(5)

Distribution \( T(x_0,x) \in S'(\mathbb{R}_1 \times \mathbb{R}^n) \) has the property of the separability of variables, if it can be represented in form 
\[
T(x_0,x) = \sum_{i} \phi_i(x_0,\psi_i(x))
\]
where distributions \( \phi(x_0) \in S'(\mathbb{R}_1) \) and \( \psi \in S(\mathbb{R}^n) \), \( x_0 \in \mathbb{R}_1 \) and \( x \in \mathbb{R}^n \) for different values of \( i \) do not depend on each other, [15].

Distribution \( T(x_0,x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) is homogeneous and of order \( \alpha \) at variable \( x_n \) and \( x \in \mathbb{R}^n \) if for \( k > 0 \) it applies that \( T(kx_0, kx) = k^\alpha T(x_0, x) \), [1, 3, 7, 8].

In other words, distribution \( T(x_0,x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) is homogeneous and of order \( \alpha \) at variable \( x_n \) and \( x \) if for each test function \( \phi(x_0,x) \in S(\mathbb{R}_1^+ \times \mathbb{R}^n) \) the following is valid
\[
\langle T(x_0,x), \phi \left( \frac{x_0}{k}, \frac{x}{k} \right) \rangle = k^{2\alpha+n+1} \langle T(x_0,x), \phi(x_0,x) \rangle,
\]
\( k > 0 \). Indeed,
\[
\langle T(kx_0, kx), \phi(x_0, x) \rangle = \left\{ \begin{array}{l}
\text{shift } kx_0 = x'_0 \Rightarrow x_0 = \frac{x'_0}{k} \\
kx = x' = x = \frac{x'}{k}
\end{array} \right.
\]
\[= \frac{1}{k^{n+1}} \langle T(x'_0, x'), \phi \left( \frac{x'_0}{k}, \frac{x'}{k} \right) \rangle = \left( \frac{x'_0}{k} = x_0, \frac{x'}{k} = x \right)
\]
\[= \frac{1}{k^{n+1}} \langle T(x_0, x), \phi \left( \frac{x_0}{k}, \frac{x}{k} \right) \rangle
\]  
(6)

For example, let it be that \( T(x_0,x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) in the form of
\[
\langle T(kx_0, kx), \phi(x_0, x) \rangle = \left( f(kx_0) \times g(kx), \phi(x_0, x) \right)
\]
\[= \left( f(kx_0), (g(kx), \phi(x_0, x)) \right) = \frac{\left( f(kx_0) = k^\alpha f(x_0) \right) \left( g(kx) = k^\alpha g(x) \right)}{\left( f(x_0) \times g(x), \phi(x_0, x) \right)}
\]
\[= k^{2\alpha} \langle f(x_0), (g(x), \phi(x_0, x)) \rangle
\]
\[= k^{2\alpha} \langle T(x_0, x), \phi(x_0, x) \rangle.
\]  
(7)

From (6) and (7) we can see that the following equation is valid
\[
\frac{1}{k^{n+1}} \langle T(x_0, x), \phi \left( \frac{x_0}{k}, \frac{x}{k} \right) \rangle = k^{2\alpha} \langle T(x_0, x), \phi(x_0, x) \rangle
\]
and from here, there is
\[
\langle T(x_0, x), \phi \left( \frac{x_0}{k}, \frac{x}{k} \right) \rangle = k^{2\alpha+n+1} \langle T(x_0, x), \phi(x_0, x) \rangle.
\]  
(8)

For example, distribution \( T(x_0,x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) in the form of
\[
\langle T(x_0,x), f(x_0) \times g(x) \rangle = \sum_{i} \phi_i(x_0, \psi_i(x))
\]
with \( f(x_0) \in S(\mathbb{R}_1^+) \), \( g(x) \in S'(\mathbb{R}_1^+) \) and \( \phi \) being homogenous and of order \( \alpha \) and test function \( \phi(x_0, x) \) in the form of
\[
\phi(x_0,x) = \sum_{i} \phi_i(x_0, \psi_i(x))
\]  
with \( \psi_i(x_0) \in S(\mathbb{R}_1^+) \), \( \psi_i(x) \in S'(\mathbb{R}_1^+) \) and \( \phi \), is a homogenous distribution of order \( \alpha \) and then, for \( (\forall i) \) the following is true
\[
\langle T(kx_0, x), f(x_0) \times g(x) \rangle = \sum_{i} \phi_i(kx_0, \psi_i(x))
\]
\[
= \sum_{i} \left( f(kx_0), \phi_i(x_0) \right) \left( g(x), \psi_i(x) \right)
\]
\[= k^{2\alpha} \sum_{i} \left( f(x_0), \phi_i(x) \right) \left( g(x), \psi_i(x) \right)
\]
\[= k^{\alpha} \sum_{i} \left( f(x_0), \phi_i(x) \right) \left( g(x), \psi_i(x) \right)
\]
\[= k^{\alpha} \langle f(x_0), g(x), \phi(x_0, x) \rangle = k^{\alpha} \langle T(x_0, x), \phi(x_0, x) \rangle.
\]  
(9)

Homogeneity at variable \( x \) is similarly observed.

Let there be distribution \( T(x_0, x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \). Distribution \( T(x_0, x) \) with \( x_0 \in \mathbb{R}_1^+ \) and \( x \in \mathbb{R}^n \) has quasi-asymptotic at infinity at variable \( x_0 \) relative to auto-modal function \( g \), if there is distribution \( G(x_0, x) \neq 0 \) such that
\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \langle T(kx_0, x), \phi(x_0, x) \rangle = G(x_0, x) \text{ in } S'(\mathbb{R}_1^+ \times \mathbb{R}^n).
\]  
(10)

Quasi-asymptotics by the separated variable at zero is similarly defined, [1, 7].

Let us suppose a distribution \( T(x_0, x) \in S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \). Distribution \( T(x_0, x) \) with \( x_0 \in \mathbb{R}_1^+ \) and \( x \in \mathbb{R}^n \) has quasi-asymptotics at zero at variable \( x_0 \) with respect to auto-modal function \( g \), if, and only if there is distribution \( G(x_0, x) \neq 0 \) such that
\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \langle T \left( \frac{x_0}{k}, x \right) \rangle = G \left( \frac{x_0}{k}, x \right) \neq 0
\]  
(11)

in \( S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \).

For distributions from \( D'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) (or \( S'(\mathbb{R}_1^+ \times \mathbb{R}^n) \) we define the fractional (rational) differentiation at variable \( x_0 \) as a convolution \( f_\alpha(x_0) \) with \( f(x_0) \) at \( x_0 \) by the following formula
\[
f^{(\alpha)} = f_{-\alpha}(x_0) \ast f(x_0)
\]  
(11)
which belongs to \( D'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) if \( f \in D'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) that is, \( S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) if \( f \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \), (more in [3-6, 8, 12, 13]).

2. Some Quasi-Asymptotics Properties of Multidimensional Distributions

We provide proof of some of the basic theorems that apply to multidimensional distributions, and their formulaic presentation can be seen in [1].

Theorem 1. If distribution \( T(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) is asymptotically homogeneous with respect to positive function \( \rho(\alpha) \) at variable \( x_0 \) or if the following is true

\[
\lim_{k \to \infty} \frac{1}{\rho(\alpha)} T(k x_0, \alpha) = G(x_0, \alpha) \quad (12)
\]

then \( \rho(\alpha) \) is an auto-modal function.

Proof: Let (12) be true and let \( \phi(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) test function such that \( (G(x_0, \alpha), \phi(x_0, \alpha)) \neq 0 \).

Then let the test function be of the following form

\[
\phi(x_0, \alpha) = \sum_i \phi_i(x_0) \varphi_i(x),
\]

so that \( \varphi_i(x_0) \in S'(\mathbb{R}_+^1), \varphi_i(x) \in S(\mathbb{R}^n) \) \( \forall i \), are continuous functions with the following feature:

\[
supp \varphi_i \subset \mathbb{R}_+^1, supp \varphi_i^2 \subset \mathbb{R}^n,
\]

\[
supp \phi = \sup sup \varphi_i^2 \times supp \varphi_i \subset (\mathbb{R}_+^1 \times \mathbb{R}^n), (\forall i),
\]

\[
K \subset \mathbb{R}_+^1 \text{ compact set.}
\]

For \( \phi(x_0, \alpha) \) and \( \alpha \in K \) it applies that

\[
\frac{1}{\alpha} \phi \left( \frac{x_0}{\alpha}, x \right) = \frac{1}{\alpha} \sum_i \phi_i \left( \frac{x_0}{\alpha}, \varphi_i(x) \right).
\]

Now, the following is valid for distribution \( T(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) and test function

\[
\phi(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n);
\]

\[
(T(k x_0, \alpha), \frac{1}{\alpha} \phi \left( \frac{x_0}{\alpha}, x \right)) \xrightarrow[k \to \infty]{\rho(\alpha)} (G(x_0, \alpha), \frac{1}{\alpha} \phi \left( \frac{x_0}{\alpha}, x \right))
\]

For \( \alpha \in K \), and using \( (x_0 = \alpha x_0) \) the following is valid

\[
\frac{\rho(\alpha)}{\rho(K)} T(k x_0, \alpha) \xrightarrow[k \to \infty]{\rho(\alpha)} G(x_0, \alpha, \phi \left( \frac{x_0}{\alpha}, x \right))
\]

Using relations (12) and (13), we get the following relation

\[
\frac{\rho(\alpha)}{\rho(K)} \xrightarrow[k \to \infty]{\rho(\alpha)} \frac{(G(x_0, \alpha), \phi \left( \frac{x_0}{\alpha}, x \right))}{\rho(K)}
\]

From here, by inserting \( (\alpha x_0 = x_0) \) we get the following

\[
\frac{\rho(\alpha)}{\rho(K)} \xrightarrow[k \to \infty]{\rho(\alpha)} \frac{(G(ax_0', \alpha), \phi(x_0', \alpha))}{\rho(K)}
\]

From here, we get the required relation

\[
\frac{\rho(\alpha)}{\rho(K)} \xrightarrow[k \to \infty]{\rho(\alpha)} \frac{(G(ax_0', \alpha), \phi(x_0', \alpha))}{\rho(K)} = C(a).
\]

From the existence of the limit \( k \to \infty \frac{\rho(\alpha)}{\rho(K)} = C(a) \) following \( C(a) = a^a \) and \( \rho(\alpha) = a^a L(a) \), and Karamata L function [16], it follows that function \( \rho(\alpha) \) is an auto-modal function, even in the case of multi-variable distributions.

Theorem 2. Let distribution \( T(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) be asymptotically homogeneous with respect to positive function \( \rho(\alpha) \) at variable \( x_0 \). In this case, if the order of auto-modal function \( \rho(\alpha) \) is equal to \( \alpha \), then distribution \( G(x_0, \alpha) \) in the following equation

\[
\frac{\rho(\alpha)}{\rho(K)} \xrightarrow[k \to \infty]{\rho(\alpha)} \frac{(G(ax_0', \alpha), \phi(x_0', \alpha))}{\rho(K)} = C(a).
\]

For the existence of the limit \( k \to \infty \frac{\rho(\alpha)}{\rho(K)} = C(a) \) following \( C(a) = a^a \) and \( \rho(\alpha) = a^a L(a) \), and Karamata L function [16], it follows that function \( \rho(\alpha) \) is an auto-modal function, even in the case of multi-variable distributions.

For \( G(x_0, \alpha) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) let us suppose that \( f(x_0) \in S'(\mathbb{R}_+^1) \), and \( g(x) = S'(\mathbb{R}^n) \), and that distribution \( f(x_0) \) is homogeneous and of order \( \alpha \), and \( G(x_0, \alpha) = f(x_0) \times g(x) \). Since for the function in the form of \( \phi(x_0, \alpha) \psi(x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n) \) the following applies

\[
\lim_{k \to \infty} \frac{1}{\rho(k)} \left( T(x_0, \alpha), \phi(x_0, \alpha) \psi(x) \right) = (G(x_0, \alpha), \phi \left( \frac{x_0}{\alpha}, x \right))
\]

so distribution \( g(x) \) is \( S'(\mathbb{R}^n) \), \( G(x_0, \alpha) \) is in the form of \( G(x_0, \alpha) = \frac{1}{\rho(K)} \frac{(G(x_0, \alpha), \phi \left( \frac{x_0}{\alpha}, x \right))}{\rho(K)} \).

Theorem 3. If distribution \( T(x_0, \alpha) \) is separated at variable forms \( x_0 \), then it has the following form:

\[
T(x_0, \alpha) = T_1 g_1(x_0) + T_2 g_2(x_0),
\]

and distribution \( T(x_0, \alpha) \) has the quasi-asymptotics of order \( \alpha \) in relation to function \( \alpha \rho(\alpha) \) at a variable \( X_0 \), if \( T_1 \) and \( T_2 \) have the same quasi-asymptotics in relation to function \( \rho(\alpha) \). The reverse of the theorem is not valid.

Proof. Let us show that distribution \( T(x_0, \alpha) \) is quasi-asymptotics of order \( \alpha \) with respect to \( \rho(\alpha) \) if \( T_1 \) and \( T_2 \) have the same quasi-asymptotics. Let the test function \( \phi(x_0, \alpha) \) be in the form of \( \phi(x_0, \alpha) = \sum_i \phi_i(x_0) \psi_i(x) \). By the definition of quasi-asymptotics, the following applies:
\[
\frac{1}{\rho(k)} (T(kx_0, x), \phi(x_0, x)) = \langle \frac{T(kx_0, x)}{\rho(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \rangle \\
= \frac{\sum_i \varphi_i(x_0) \psi_i(x)}{k^a \rho(k)} \\
= \frac{\sum_i \varphi_i(x_0) \psi_i(x)}{\rho(k)} + \frac{T_1(kx_0, g_1(x)) + T_2(kx_0, g_2(x))}{k^a \rho(k)} \sum_i \varphi_i(x_0) \psi_i(x). \]

Since \( \rho(k) = k^a L(k) \) and \( T_1(kx_0) = k^a T_1(x_0) \) and \( T_2(kx_0) = k^a T_2(x_0) \) therefore

\[
= \left( \frac{T_1(kx_0, g_1(x))}{k^a L(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) + \left( \frac{T_2(kx_0, g_2(x))}{k^a L(k)}, \sum_i \varphi_i(x_0) \psi_i(x) \right) \\
= \frac{1}{k^a L(k)} \sum_i \left( T_1(kx_0, \varphi_i(x_0)) \langle g_1(x), \psi_i(x) \rangle \right) + \frac{1}{k^a L(k)} \sum_i \left( T_2(kx_0, \varphi_i(x_0)) \langle g_2(x), \psi_i(x) \rangle \right) \\
= \frac{k^a}{k^a L(k)} \sum_i \left( T_1(kx_0, \varphi_i(x_0)) \langle g_1(x), \psi_i(x) \rangle \right) + \frac{k^a}{k^a L(k)} \sum_i \left( T_2(kx_0, \varphi_i(x_0)) \langle g_2(x), \psi_i(x) \rangle \right) \\
= \frac{1}{L(k)} \sum_i \left( T_1(kx_0, \varphi_i(x_0)) \langle g_1(x), \psi_i(x) \rangle \right) + \frac{1}{L(k)} \sum_i \left( T_2(kx_0, \varphi_i(x_0)) \langle g_2(x), \psi_i(x) \rangle \right) \\
= \left( \frac{T_1(kx_0, g_1(x))}{L(k)}, \varphi(x_0, x) \right) + \left( \frac{T_2(kx_0, g_2(x))}{L(k)}, \varphi(x_0, x) \right) \\
= \left( \frac{T_1(kx_0, g_1(x)) + T_2(kx_0, g_2(x))}{L(k)}, \varphi(x_0, x) \right). \]

This shows that distribution \( T(x_0, x) = T_1 g_1(x) + T_2 g_2(x) \) has quasi-asymptotics of order \( \alpha \) with respect to function \( k^a \rho(k) \) at variable \( x_0 \) if distributions \( T_1 \) and \( T_2 \) have the same quasi-asymptotics.

The reverse of the theorem is not valid. To show this, it is enough to show that, for example, the following is not valid for distribution \( T_1(x_0) = x_0^{a+1} + x_0^a \) and \( T_2(x_0) = -x_0^{a+1} + x_0^a \) with respect to function \( k^a \rho(k) \).

Theorem 4. In order for distribution \( T(x_0, x) \in S'(\mathbb{R}_+^n \times \mathbb{R}^n) \) to be asymptotically homogeneous at infinity, with respect to auto-modal function \( \rho(k) \) at variable \( x_0 \), it is necessary, and it is also sufficient, that for each \( \beta \in \mathbb{R} \) its...
fractional derivative $T^{(-\beta)}(x_0, x)$ is asymptotically homogeneous with respect to $k^{\beta} \rho(k)$.

Proof: We define fractional differentiation in $S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ with distribution $T(x_0, x)$ at $x_0$ as convolution of distribution $f_\beta(x_0) \in S'(\mathbb{R}_+^1)$ and distribution $T(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n)$ i.e. $T^{(-\beta)}(x_0, x) = T(x_0, x) * f_\beta(x_0)$. Using the property of distribution $f_\beta(x_0)$ to be homogeneous and of order $\beta - 1$, that is, using the validity of the following $f_\beta(x_0) = k^{\beta-1} f_\beta(x_0)$, we get the following:

$$\lim_{k \to \infty} \frac{1}{k^{\beta-1} \rho(k)} (T^{(-\beta)}(x_0, x), \phi(x_0, x))$$
from here, if we put that, $kx_0 = x'$ we get

$$= \lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T^{(-\beta)}(x', x), \phi\left(\frac{x'}{k}, x\right))$$

$$= \lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T^{(-\beta)}(x_0, x), \phi\left(\frac{x_0}{k}, x\right)).$$

By using the definition of convolution

$$T(x_0, x) * f_\beta(x_0) = \frac{1}{\Gamma(\beta)} \Theta(x_0)x_0^{\beta-1} * T(x_0, x)$$

$$= \frac{1}{\Gamma(\beta)} \int_0^{\infty} (x_0, x_0)^{\beta-1} T(t, x) dt = T^{(-\beta)}(x_0, x).$$

we can see that the last equation is precisely the $\beta$ primitive integral for $T(x_0, x)$. Based on this, we have that $T(x_0, x) \in S'(\mathbb{R}_+^1 \times \mathbb{R}^n), f_\beta(x_0) \in S'(\mathbb{R}_+^1)$,

$$\langle T(x_0, x) * f_\beta(x_0), \phi(x_0, x) \rangle$$

$$= \lim_{k \to \infty} \langle T(x_0, x) * f_\beta(x_0), \phi(x_0, x) \rangle,$$

with $\{\eta_k\}$ being unit sequence. If there is a limes on the right-hand side for each series $\eta_k, k \to \infty$ then the function from $S'(\mathbb{R}^2)$ which converges to number one in $S^2$ and this limit does not depend on the choice of series $\{\eta_k, k \to \infty\}$ then we have that $T(x_0, x) * f_\beta(x_0) \in S'(\mathbb{R}^{n+1})$. Based on this, the last equation transforms into

$$\lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T(x_0, x) * f_\beta(x_0), \phi\left(\frac{x_0}{k}, x\right))$$

$$= \lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T(x_0, x) * f_\beta(x_0), \phi\left(\frac{x_0}{k}, x\right)).$$

Now, if we put that

$$\left(\frac{x_0 + \tau}{k} = \frac{x_0 + k \tau}{k} = \frac{x_0}{k} + \frac{\tau}{k}\right)$$

the last equation transforms into the following form:

$$\lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T(x_0, x), \phi\left(\frac{x_0}{k}, x\right))$$

and

$$\lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T(x_0, x), \phi\left(\frac{x_0}{k}, x\right)).$$

(since $f_\beta(k \tau) = k^{\beta-1} f_\beta(\tau)$)

$$\lim_{k \to \infty} \frac{k^{\beta-1}}{k^{\beta+1} \rho(k)} (T(x_0, x), \phi\left(\frac{x_0}{k}, x\right)).$$

From the last equation, using the shift $(x_0 = k x')$ we get the following

$$\lim_{k \to \infty} \frac{1}{k^{\beta+1} \rho(k)} (T(x_0, x), \phi(x_0, x)).$$

3. Example of the Use of Quasi-Asymptotics to the Solutions of Differential Equations

Let $L$ be a differential operator with constant coefficients $a_\beta(x) = a_\beta$ and let $f \in \mathcal{D}'$, be such a distribution that convolution $E * f$ exists in $\mathcal{D}'$ where $E \in \mathcal{D}'$ is the fundamental solution of equation $L(D)E = \delta(x)$, $[3, 6, 9, 11]$.

Then the solution $u = E * f$ of differential equation $L(D)u = f(x), f \in \mathcal{D}'$ has quasi-asymptotics of order $a$ with respect to $\rho(k) = k^a L(k)$ (with $L(k)$ being the Karamata slow-varying function), if distribution $f \in \mathcal{D}'$ has such quasi-asymptotics, $\mathcal{D}'$-distribution space.

Proof: Let $f$ have the quasi-asymptotics with respect to $\rho(k) = k^a L(k)$. Then the following is valid

$$\frac{1}{\rho(k)} (f(kx), \phi(x)) = \frac{1}{k \rho(k)} (f(x), \phi\left(\frac{x}{k}\right))$$
\[
\frac{1}{k \rho(k)} \left( \delta(x) * f(x), \varphi \left( \frac{x}{k} \right) \right)
\]

\[
\frac{1}{k \rho(k)} \left( L(D)E * f(x), \varphi \left( \frac{x}{k} \right) \right)
\]

\[
= \frac{1}{k \rho(k)} \left( \sum_{|a|=0}^{m} a_{a}D^{a}E(x) * f(x), \varphi \left( \frac{x}{k} \right) \right)
\]

\[
= \frac{1}{k \rho(k)} \left( L(D)(E * f)(x), \varphi \left( \frac{x}{k} \right) \right)
\]

Therefore, we have the following:

\[
\frac{1}{\rho(k)}(f(kx), \phi(x)) = (\frac{u(kx)}{\rho(k)}, L(-D) \phi(x)), \quad \text{and, as per assumption, } f \text{ has the quasi-asymptotics, thus, distribution } u
\]

has one also.

4. Conclusion

Most of the theorems proved in this paper on quasi-asymptotics of distributions at a separable variable have their analog in the case of one-dimensional distributions. In [1], Vladimirov showed a theorem that does not have a one-dimensional analog, the consequence of which is very important, and on the basis of which the application of separated quasi-asymptotics in to the solutions of differential equations.

References


