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Continuous Explicit Hybrid Method for Solving Second Order Ordinary Differential Equations

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Abstract: This paper presents an explicit hybrid method for direct approximation of second order ordinary differential equations. The approach adopted in this work is by interpolation and collocation of a basis function and its corresponding differential system respectively. Interpolation of the basis function was done at both grid and off-grid points while the differential systems are collocated at selected points. Substitution of the unknown parameters into the basis function and simplification of the resulting equation produced the required continuous, consistent and symmetric explicit hybrid method. Attempts were made to derive starting values of the same order with the methods using Taylor's series expansion to circumvent the inherent disadvantage of starting values of lower order. The methods were applied to solve linear, non-linear, Duffing equation and a system of equation second-order initial value problems directly. Errors in the results obtained were compared with those of the existing implicit methods of the same and even of higher order. The comparison shows that the accuracy of the new method is better than the existing methods.

Keywords: Continuous, Duffing Equation, Explicit, Symmetric, Zero-stable, Approximate Solution, Power Series

1. Introduction

Many principles, or natural laws, underlying the behavior of many phenomena in the world are statements or relations involving rates at which things happened. When expressed in mathematical models the relations are equations and the rates are derivatives. According to Khlebopros *et al.* [1], mathematical modeling is a key tool for the analysis of a wide range of real-world problems ranging from physics and engineering to chemistry, biology and even economics using differential equations. Many of these mathematical models result into differential equations of high order. The solution of many of the equations could not be obtained analytically, hence the approximate solution by different numerical methods. These equations have been considered by several researchers and have been numerically approximated directly by circumventing the conventional method of reducing to system of first order equations before adopting appropriate method to solve them, see [2-5]. The main aim of developing numerical methods in recent times is to present numerical methods with an improved level of accuracy Adeyeye and Omar [5].

Various authors such as Liu and Jhao [6] investigated the solution of the nonlinear Duffing oscillator using a modified power series expansion method. This method is suitable for a long term computation of nonlinear ODEs. Yusuf *et al.* [7] proposed a fifth order zero-dissipative trigonometrically fitted two-step hybrid method for solving oscillatory problems. The coefficient of the method according to the paper depends on the frequency of the problem to be solved. While Olabode and Momoh [8] constructed a continuous hybrid method for the solution of second order stiff ODEs which was implemented in block mode. More recently, [9] investigated a hybrid multistep method for numerical solution of special second order initial value problems. The method catered for the elimination of phase-lag and amplification error due to addition of multi-free parameters. However, researchers have also looked into implementing derivative methods in solving ODEs, [10, 11]. Also in 2019, Singh and Ramos [12] presented an optimized two-step hybrid block method which is formulated in variable step size mode to solve (1). In 2013, Kayode and Obarhua [13] presented a continuous $y-function$ hybrid method for direct solution of second order IVPs of ODEs. In the paper, the explicit method of same order of accuracy with the method was used as the main predictor for the implementation of the method.

In this work, therefore, it is of the interest to investigate a continuous explicit method (main predictor in [13]) for the solution of general second order ordinary differential equations for the purpose of enhancing and comparing the accuracy with existing methods.

2. Materials and Method

In this paper the solution of second order initial value problem of the form
$y''=f(x,y,y'),\ y(x_0)=y_0,\ y'(x_0)=y_1,$ is considered, where

$$x_0, y_0, y_1 \in \mathbb{R} \tag{1}$$

The method for the solution of (1) is developed by using a partial sum of power series as an approximate solution to the problem as

$$y(x)=\sum_{j=0}^{3k+1} a_j x^j \tag{2}$$

where $a_j=\{\mathbb{R}\}$ for $j=1,\cdots,3k+1$, where k is fixed in $\mathbb{N}$.

Taking the first and second derivatives of equation (2) to have

$$y'(x)=\sum_{j=1}^{3k+1} j a_j x^{j-1} \tag{3}$$

$$y''(x)=\sum_{j=2}^{3k+1} j(j-1) a_j x^{j-2} \tag{4}$$

Combining equations (1) and (4) to have

$$\sum_{j=2}^{3k+1} j(j-1) a_j x^{j-2} = f(x, y(x), y'(x)) \tag{5}$$

By interpolating equation (2) at $x_{n+\zeta},\ \zeta=0\left(\frac{1}{2}\right)\frac{3}{2}$ and collocating (5) at $x_{n+\zeta},\ \zeta=0,1,\frac{3}{2}$ the following system of linear equations are obtained to be

$$\begin{aligned}&\sum_{j=0}^{3k+1} a_j x_{n+\zeta}^j = y_{n+\zeta},\ \zeta=0\left(\frac{1}{2}\right)k-1,\\ &\sum_{j=2}^{3k+1} j(j-1) a_j x_{n+\zeta}^{j-2} = f_{n+\zeta},\ \zeta=0,1,k-\frac{1}{2}\end{aligned} \tag{6}$$

where $y_{n+\zeta}$ represents the approximate solution $y(x)$ at $x_{n+\zeta}$ and $f_{n+\zeta}=f(x_{n+\zeta}, y_{n+\zeta}, y'_{n+\zeta})$.
for $k=2$, the system of equations (6) are solved for the unknown parameters $a_j,\ j=0(1)7$. Solving this system of equation (6) for $k=2$ for the unknown parameters $a_j's,\ j=0(1)7$. The values of these parameters are substituted into the approximate solution (2) and using transformations in Kayode and Obarhua [14], the following are obtained:

$$\begin{aligned}
a_0 &= y_n,\\
a_1 &= -\frac{149}{21}y_n+\frac{72}{7}y_{n+\frac{1}{2}}-\frac{9}{7}y_{n+1}-\frac{40}{21}y_{n+\frac{3}{2}}+\frac{h^2}{70}\left(-2f_n+66f_{n+\frac{1}{2}}+39f_{n+1}+2f_{n+\frac{3}{2}}\right),\\
a_2 &= \frac{h^2}{2}f_n,\\
a_3 &= 64y_n-112y_{n+\frac{1}{2}}+32y_{n+1}+16y_{n+\frac{3}{2}}+\frac{h^2}{18}\left(-35f_n-228f_{n+\frac{1}{2}}-93f_{n+1}-4f_{n+\frac{3}{2}}\right),\\
a_4 &= -\frac{400}{3}y_n+\frac{760}{3}y_{n+\frac{1}{2}}-\frac{320}{3}y_{n+1}-\frac{40}{3}y_{n+\frac{3}{2}}+\frac{h^2}{9}\left(28f_n+245f_{n+\frac{1}{2}}+56f_{n+1}+f_{n+\frac{3}{2}}\right)\\
a_5 &= 112y_n-240y_{n+\frac{1}{2}}+144y_{n+1}-16y_{n+\frac{3}{2}}+\frac{h^2}{5}\left(-12f_n-114f_{n+\frac{1}{2}}+4f_{n+1}+2f_{n+\frac{3}{2}}\right),\\
a_6 &= -\frac{128}{3}y_n+\frac{320}{3}y_{n+\frac{1}{2}}-\frac{256}{3}y_{n+1}+\frac{64}{3}y_{n+\frac{3}{2}}+\frac{h^2}{9}\left(8f_n+76f_{n+\frac{1}{2}}-32f_{n+1}-4f_{n+\frac{3}{2}}\right),\\
a_7 &= \frac{128}{21}y_n-\frac{128}{7}y_{n+\frac{1}{2}}+\frac{128}{7}y_{n+1}-\frac{128}{21}y_{n+\frac{3}{2}}+\frac{h^2}{63}\left(-8f_n-72f_{n+\frac{1}{2}}+72f_{n+1}+8f_{n+\frac{3}{2}}\right).
\end{aligned} \tag{7}$$

Substituting (7) into approximate equation (2), the following continuous coefficients hybrid methods are derived:

$$y_k(t)=\sum_{j=0}^{k-1}\alpha_j(t)y_{n+j}+h^2\sum_{j=0}^{k}\beta_j(t)f_{n+j},\ \beta_k=0,\ for\ \ t\in(0,1]\,. \tag{8}$$

Applying the transformation $t=\frac{1}{h}(x-x_{n+k-1}),\ dt=\frac{1}{h}dt,\ t\in(0,1]$ in [14] to (8) its continuous coefficients α_j's and β_j's are obtained as

$$\alpha_0(t)=\frac{1}{21}(128t^7-336t^5+224t^3-37t),$$

$$\alpha_{\frac{1}{2}}(t)=\frac{1}{21}(-384t^7-448t^6+336t^5+280t^4-112t^3-8t),$$

$$\alpha_1(t)=\frac{1}{21}(384t^7+896t^6+336t^5-560t^4-448t^3+85t+21),$$

$$\alpha_{\frac{3}{2}}(t)=\frac{1}{21}(-128t^7-448t^6-336t^5+280t^4+336t^3-40t),$$

$$\beta_0(t)=\frac{1}{630}(-80t^7+168t^5-105t^3+17t),$$

$$\beta_{\frac{1}{2}}(t)=\frac{1}{315}(-360t^7+140t^6+1218t^5-35t^4-910t^3+157t),$$

$$\beta_1(t)=\frac{1}{3150}(3600t^7+14000t^6+10920t^5-9800t^4-10675t^3+1575t^2+1930t),$$

$$\beta_{\frac{3}{2}}(t)=\frac{1}{630}(80t^7+280t^6+252t^5-70t^4-140t^3+18t). \tag{9}$$

The first derivatives of the coefficients are

$$\alpha_0'(t)=\frac{1}{21}(896t^6-1680t^4+672t^2-37),$$

$$\alpha_{\frac{1}{2}}'(t)=\frac{1}{21}(-2688t^6-2688t^5+1680t^4+1120t^3-336t^2-8),$$

$$\alpha_1'(t)=\frac{1}{21}(2688t^6+5376t^5+1680t^4-2240t^3-1344t^2+85),$$

$$\alpha_{\frac{3}{2}}'(t)=\frac{1}{21}(-896t^6-2688t^5-1680t^4+1120t^3+1008t^2-40),$$

$$\beta_0'(t)=\frac{1}{630}(-560t^6+840t^4-315t^2+17),$$

$$\beta_{\frac{1}{2}}'(t)=\frac{1}{315}(-2520t^6+840t^5+6090t^4-140t^3-2730t^2+157),$$

$$\beta_1'(t)=\frac{1}{3150}(25200t^6+84000t^5+54600t^4-39200t^3-32025t^2+3150t+1930),$$

$$\beta_{\frac{3}{2}}'(t)=\frac{1}{315}(280t^6+840t^5+630t^4-140t^3-210t^2+9). \tag{10}$$

Evaluating equations (9) and (10) at $t=1$, produced discrete explicit hybrid method and its first derivative respectively.

$$y_{n+2} = -16y_{n+\frac{3}{2}} + 34y_{n+1} - 16y_{n+\frac{1}{2}} - y_n + \frac{h^2}{3}\left(2f_{n+\frac{3}{2}} + 11f_{n+1} + 2f_n\right), \tag{11}$$

$$y'_{n+2} = \frac{1}{21h}\begin{pmatrix} -3176y_{n+\frac{3}{2}} + 6245y_{n+1} - 2920y_{n+\frac{1}{2}} - \\ 149y_n \end{pmatrix} + \frac{h}{630}\begin{pmatrix} 2818f_{n+\frac{3}{2}} + 19531f_{n+1} + 3394f_{n+\frac{1}{2}} - \\ 18f_n \end{pmatrix}. \tag{12}$$

The method (11) and its derivative (12) above are of order six. Their error constants c_{p+2} are 2.3768×10^{-5} and 2.2308×10^{-4} respectively.

3. Implementation of the Method

In this section, the explicit method (11) is implemented by solving some test problems. The starting values for $y_{n+i},\ i=\frac{1}{2}\left(\frac{1}{2}\right)\frac{3}{2}$, are obtained by using Taylor series expansion

$$y_{n+i} = y_n + \frac{h}{2}y'_n + \frac{(h/2)^2}{2!}f_n + \frac{(h/2)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + \cdots + o(h^6) \tag{13}$$

and

$$y'_{n+i} = y'_n + \frac{h}{2}f_n + \frac{h^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + \cdots + o(h^6) \tag{14}$$

4. Test Problems

The developed method is applied to solve undamped Duffing equation, non-linear, linear and system of equations of second order initial value problems which were solved in [6], [13] and [15] respectively using Maple code to show the accuracy of the developed Explicit Hybrid Method (EHM) and the results are shown in Tables 1 – 4. The approximate solutions with the EHM are compared with the approximate solutions obtained with different methods in the literature. The following notations are used in the tables.

TOL-Tolerance

MTD-Method employed

TS-Total steps taken

MAXE-Magnitude of the maximum error of the computed solution

t_e -The execution time taken in microseconds

2PFDIR-Direct two point two step block method of variable step size in [15]

2-STEP-Two step method with hybrid pints $\frac{1}{2}$ and $\frac{3}{2}$

y_{exact} - Exact solution

$y_{computed}$ - Numerical solution

Problem 1: (The Undamped Duffing Equation)

$$y'' + \alpha y + \beta y^3 = 2\sin x - 2\sin^3 x,\ y(0)=1,\ y'(0)=\frac{1}{2},$$
$$\alpha = 3,\ \beta = -2,\ \ x\in[0,1], h=0.1$$

The theoretical solution is

$$y(x) = \sin x$$

Table 1. *The absolute errors* $\left|y_{exact} - y_{computed}\right|$ *obtained with the method for Problem 1 is compared with that of [6].*

x	y_{exact}	$y_{computed}$	Errors in [6], for Problem 1	Errors in New Scheme (11) for Problem k=2, p=6
0.1	0.099833416646828155	0.099833416646828154	3.08320e-018	1.32156e-019
0.2	0.198669330795061220	0.198669330795061293	9.94755e-018	7.31247e-018
0.3	0.295520206661339600	0.295520206661339624	1.58294e-017	2.46001e-018
0.4	0.389418342308650520	0.389418342308650567	1.03541e-017	4.75722e-018
0.5	0.479425538604203010	0.479425538604203009	5.09575e-018	1.00147e-018
0.6	0.564642473395035480	0.564642473395035460	3.24719e-017	2.04587e-018
0.7	0.644217687237691130	0.644217687237691120	2.87314e-018	1.04314e-018
0.8	0.717356090899522790	0.717356090899522772	3.48571e-018	1.81124e-018
0.9	0.783326909627483410	0.783326909627483402	3.00324e-017	8.56714e-018
1.0	0.841470984807896500	0.841470984807896453	5.28006e-017	4.70129e-017

Problem 2: (Non-Linear Problem)

$$y'' = x(y')^2,\ y(0) = 1,\ y'(0) = \frac{1}{2},\ h = \frac{1}{30}.$$

The theoretical solution is

$$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right).$$

Table 2. *The absolute errors* $\left|y_{exact} - y_{computed}\right|$ *obtained with the method for Problem 2 is compared with that of [13] 2-step continuous* $y-function$ *method of same order of accuracy.*

x	y_{exact}	$y_{computed}$	**Errors in [13], for Problem 2** $k=2,\ p=6$	**Errors in New Scheme (11) for Problem 2** $k=2, p=6$
0.1	1.05004172927849	1.05004172933139	2.806096e-09	5.29040e-10
0.2	1.10033534773107	1.10033534754502	1.568605e-08	1.86050e-10
0.3	1.15114043593646	1.15114043223415	4.021738e-08	3.70231e-09
0.4	1.20273255405408	1.20273255123188	7.887442e-08	2.82222e-09
0.5	1.25541281188299	1.25541281315673	1.357736e-07	1.27374e-09

Problem 3: (Linear Problem)

$$y'' + \frac{6}{x}y' + \frac{4}{x^2}y = 0,\ y(1) = 1 = y'(1),\ h = \frac{1}{320}.$$

The theoretical solution is

$$y(x) = \frac{5}{3x} - \frac{2}{3x^4}.$$

Table 3. *The absolute errors* $\left|y_{exact} - y_{computed}\right|$ *obtained with the method for Problem 3 is compared with that of [13] 2-step continuous* $y-function$ *method of same order of accuracy.*

x	y_{exact}	$y_{computed}$	**Errors in [13], for Problem 3** $k=2, p=6$	**Errors in New Scheme (11) for Problem 3** $k=2, p=6$
1.0094	1.00894499508883	1.00894499575316	9.661260e-08	1.3567e-10
1.0125	1.01174101816798	1.01174101847327	9.425732e-08	6.4125e-10
1.0156	1.01444754268641	1.01444754245054	9.197108e-08	2.3587e-09
1.0188	1.01706649423567	1.01706649390086	8.975049e-08	3.3481e-09
1.0219	1.01959975475628	1.01959975420736	8.759359e-08	5.4892e-09
1.0250	1.02204916362943	1.02204916320505	8.549846e-08	4.2438e-09
1.0281	1.02441651873840	1.02441651801149	8.346327e-08	7.2691e-09
1.0313	1.02670357750080	1.02670357684810	8.148622e-08	6.5270e-09

Problem 4: (Two body Problem)

$$y_1'' = \frac{-y_1}{r},\ \ y_1(0) = 1,\ \ y_1'(0) = 0, y_2'' = \frac{-y_1}{r},\ \ y_2(0) = 0,\ \ y_2'(0) = 1, r = \sqrt{{y_1}^2 + {y_2}^2}$$

The theoretical solution is

$$y_1(x) = \cos x, y_2(x) = \sin x$$

Table 4. *The maximum errors* $\left|y_{exact} - y_{computed}\right|$ *obtained with the method for Problem 4, the execution time in microseconds* t_e *and the total steps taken are compared with that of [15] two step four point block method.*

TOL	**Majid *et al.* [15] for Problem 4**				MTD	TS	**New Scheme for Problem 4**	
	MTD	**TS**	**MAXE**	t_e			**MAXE**	t_e
10^{-2}	2PFDIR	67	7.98175e-002	938	2-STEP	33	8.247561e-008	360
10^{-4}	2PFDIR	140	6.93117e-004	1472	2-STEP	55	7.521783e-010	1312
10^{-6}	2PFDIR	316	7.46033e-006	3318	2-STEP	74	9.286714e-012	2256
10^{-8}	2PFDIR	394	2.45673e-006	4181	2-STEP	130	1.561010e-015	2549
10^{-10}	2PFDIR	938	2.53897e-008	9932	2-STEP	274	2.204611e-017	4059

5. Conclusion

This article presents a continuous explicit method of order six for direct solution of both the undamped equation, linear, nonlinear and system of equations of second order initial value problems of ODEs. The method is consistent, convergent and zero stable. The method had served as a predictor for implementation of an implicit method in [13]. The test problems are solved and the results are tabulated.

The results in the Tables 1 – 4 are evidence that the new method compared favorably with its implicit methods in [6], [13] and [15] in terms of accuracy and efficiency.

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