Pure and Applied Mathematics Journal
2019; 8(6): 93-99
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20190806.11
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Hardy-Rogers Type Mappings for Fuzzy Metric Space

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To cite this article:
Mohit Kumar, Ritu Arora, Ajay Kumar. Hardy-Rogers Type Mappings for Fuzzy Metric Space. *Pure and Applied Mathematics Journal.* Vol. 8, No. 6, 2019, pp. 93-99. doi: 10.11648/j.pamj.20190806.11

Received: July 23, 2019; **Accepted**: September 26, 2019; **Published**: December 24, 2019

Abstract: The evolution of fuzzy mathematics commenced with the introduction of the notion of fuzzy set by Zadeh, where the concept of uncertainty has been introduced in the theory of sets in a non probabilistic manner. The several researchers were conducting the generalization of the concept of fuzzy sets. The present research paper focuses on the existence of fixed points in fuzzy metric space. Hardy-Rogers is to establish a fixed point theorem for three maps of a complete metric space. The contractive condition is generalized and the commuting condition of Jungck is replaced by the concept of weakly commuting. The three Hardy-Rogers type mappings are extended in fuzzy metric space and also extend to generalize non-expansive mapping define over a compact fuzzy metric space. The contractive condition is generalization of Hardy-Rogers and the commuting condition of Jungck is replace by the concept of weakly commuting. Our results deals with mappings satisfying a condition weaker than commutativity in complete fuzzy metric space and is the generalization in complete fuzzy metric space of Hardy-Rogers type mappings in complete metric space. We also provide some illustrative example to support our result. We apply also our main results to derive unique and common fixed point for contractive mappings.

Keywords: Weakly Commuting Mapping, Asymptotically Regular Mapping, Compact Fuzzy Metric Space, Fixed Point

1. Introduction

The concept of fuzzy sets and fuzzy logic was introduced by Professor Zadeh [8] in 1965. The several researchers were conducting the generalization of the concept of fuzzy sets. Fuzzy set theory has an application in a variety of fields such as applied sciences, image processing, artificial intelligence, computer science, control engineering, computer applications, robotics and many more.

Kramosil and Michalek [7] introduced to concept of fuzzy metric space in 1975 and Grabiec [9] has proved contraction principle in fuzzy metric spaces in 1988.

Consequently, some fixed point results were generalized to fuzzy metric space by authors as [1, 3, 9, 10, 17] etc. The result establishes a fixed point theorem for three maps of a complete metric space. The contractive definition is a generalization of that of Hardy-Rogers and the commuting condition of Jungck is replace by the concept of weakly commuting [2, 4].

Inspired by the work of Hardy-Rogers, we prove some fixed point theorems for Hardy-Rogers type mappings on complete fuzzy metric space. Our results deals with mappings satisfying a condition weaker than commutativity in complete fuzzy metric space and is the generalization in complete fuzzy metric space of Hardy-Rogers type mappings in complete metric space [2].

The aim of this paper is to prove some fixed point theorems for Hardy-Rogers type mappings on complete fuzzy metric space and also extend to generalize non-expansive mapping defined over a compact fuzzy metric space.

2. Preliminaries

1. Definition (Schweizer and Sklar [13]) A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

[B.1] $*$ is commutative and associative
[B.2] $*$ is continuous
[B.3] $a*1=a \ \ \forall \ a\in[0,1]$
[B.4] $a*b\le c*d$ whenever $a\le c, b\le d$ and $a,b,c,d\in[0,1]$.

2. Definition (George and Veeramani [1]) The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy metric in $X^2\times[0,\infty]\to[0,1]$, satisfying the following conditions: for all $x,y,z\in X$, and $t,s>0$.
   [FM.1] $M(x,y,0)=0$
   [FM.2] $M(x,y,t)=1\ \forall \ t>0$ if and only if $x=y$.
   [FM.3] $M(x,y.t)=M(y,x,t)$
   [FM.4] $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$
   [FM.5] $M(x,y,\bullet):[0,\infty]\to[0,1]$, is left continuous
   [FM.6] $\lim_{t\to\infty} M(x,y,t)=1$.
3. Definition (George and Veeramani [1]) Let $(X,M,*)$ be a fuzzy metric space and let a sequence $\{x_n\}$ in X is said to be converge to $x\in X$ if $\lim_{n\to\infty} M(x_n,x,t)=1$, for each $t>0$.
4. Definition (George and Veeramani [1]) A sequence $\{x_n\}$ in X is called Cauchy sequence if $\lim_{n\to\infty} M(x_n,x_{n+p},t)=1$, for each $t>0$, and $p=1,2,3,\cdots$
5. Definition (George and Veeramani [1]) A fuzzy metric space $(X,M,*)$ is said to be complete if every Cauchy sequence in X is convergent in X.
   A fuzzy metric space in which every Cauchy sequence is convergent is called complete. It is called compact if every sequence contains a convergent subsequence.
6. Definition (George and Veeramani [1]) A self mapping $T:X\to X$ is called fuzzy contractive mapping if $M(Tx,Ty,t)>M(x,y,t)$ for each $x\ne y\in X$, and $t>0$.
7. Definition (Jungck [4]) Two mappings f and g of a fuzzy metric space $(X,M,*)$ into itself are said to be weakly commuting maps if $M(fgx,gfx,t)\ge M(fx,gx,t)$ for each $x\in X$.
8. Definition (Mishra and Singh [14]) Let f and g be two self maps of X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be asymptotically f-regular with respect to g if $\lim_{n\to\infty} M(gx_n,fx_n,t)=1$.
9. Definition (George and Veeramani [1]) Let $(X,M,*)$ be a fuzzy metric space. The open ball $B(x,r,t)$ and closed ball $B[x,r,t]$ with centre $x\in X$ and radius r, $0<r<1$, $t>0$ respectively, are defined as follows:

$$B(x,r,t)=\{y\in X: M(x,y,t)>1-r\}$$
$$B[x,r,t]=\{y\in X: M(x,y,t)\ge 1-r\}.$$

10. Definition (Hardy and Rogers [5]) A mapping $T:X\to X$ is said to be generalized non-expansive if for all $x,y\in X$ then the inequality

$$d(Tx,Ty)\le a_1d(x,Tx)+a_2d(y,Ty)+a_3d(x,Ty)+a_4d(y,Tx)+a_5d(x,y)$$

holds, where $a_i\ge 0, i=1,\cdots,5$ and $\sum_{i=1}^{5}a_i\le 1$. Due to symmetry, one can choose $a_1=a_2$ and $a_3=a_4$.

11. Definition A mapping $P:X\to X$ is said to be generalized non-expansive if for all $x,y\in X$ then the inequality

$$M(Px,Py,t)\ge a_1M(x,Px,\frac{t}{5})+a_2M(y,Py,\frac{t}{5})+a_3M(x,Py,\frac{t}{5})+a_4M(y,Px,\frac{t}{5})+a_5M(x,y,\frac{t}{5})$$

holds, where $a_i\ge 0, i=1,\cdots,5$ and $\sum_{i=1}^{5}a_i\le 1$. Due to symmetry, one can choose $a_1=a_2$ and $a_3=a_4$.

12. Definition (Kannan [11]) A mapping T is said to have property (K) on a subset G of X if for every closed subset F of G which contains more than one point is mapped into itself by T, there exists $x\in F$ such that $d(x,Tx,)<\sup_{y\in F} d(y,Ty,)$.
13. Definition A mapping P is said to have property(M) on a subset G of X if for every closed subset F of G which contains more than one point is mapped into itself by P, there exists an $x\in F$ such that $M(x,Px,t)>\sup_{y\in F} M(y,Py,t)$.

3. Main Results

Result in Complete Fuzzy Metric Space

Theorem 3.1 Let P,Q,R be self- maps of a complete fuzzy metric space X satisfying

$$\text{(i)}\quad \begin{aligned} M(Px,Py,t)\ge\ & a_1M(Qx,Px,\frac{t}{10})+a_2M(Rx,Px,\frac{t}{10})\\ &+a_3M(Qy,Py,\frac{t}{10})+a_4M(Ry,Py,\frac{t}{10})\\ &+a_5M(Qx,Py,\frac{t}{10})+a_6M(Rx,Py,\frac{t}{10})\\ &+a_7M(Qy,Px,\frac{t}{10})+a_8M(Ry,Px,\frac{t}{10})\\ &+a_9M(Qx,Ry,\frac{t}{10})+a_{10}M(Qy,Rx,\frac{t}{10})\end{aligned}$$

where $a_i=a_i(x,y)$ are non-negative functions of x and y satisfying

$$\max\{\sup_{x,y\in X}(b_3+b_3'+b_4+b_4'+b_5+b_5'),$$

(ii) $$\sup_{x,y\in X}(b_1'+b_2+b_3+b_4'+b_5+b_5'),$$

$$\sup_{x,y\in X}(b_1'+b_2'+b_3+b_4)\}<1$$

and b_1, b_2 are bounded, then b_i, b_i' as defined in (vi)

(iii) Q and R are continuous,

(iv) $\{P,Q\}$ and $\{P,R\}$ are weakly commuting pairs, and

(v) There exists an asymptotically P-regular sequence with respect to both Q and R.

Then P,Q and R have a common unique fixed point. Further P is continuous at the fixed point if $\sup_{x,y\in X}(b_1+b_2+b_3'+b_4')<1$.

Proof Interchanging the roles of x and y and then adding the two resulting inequalities yields

(i′)

$$\begin{aligned}M(Px,Py,t) &\geq b_1(x,y)M(Qx,Px,\tfrac{t}{10})+b_1(y,x)M(Qy,Py,\tfrac{t}{10})\\ &+b_2(x,y)M(Rx,Px,\tfrac{t}{10})+b_2(y,x)M(Ry,Py,\tfrac{t}{10})\\ &+b_3(x,y)M(Qx,Py,\tfrac{t}{10})+b_3(y,x)M(Qy,Px,\tfrac{t}{10})\\ &+b_4(x,y)M(Rx,Py,\tfrac{t}{10})+b_4(y,x)M(Ry,Px,\tfrac{t}{10})\\ &+b_5(x,y)M(Qx,Ry,\tfrac{t}{10})+b_5(y,x)M(Qy,Rx,\tfrac{t}{10})\end{aligned}$$

where

$$\begin{aligned}2b_1(x,y)&=a_1(x,y)+a_3(y,x)\\ 2b_2(x,y)&=a_2(x,y)+a_4(y,x)\\ 2b_3(x,y)&=a_5(x,y)+a_7(y,x)\\ 2b_4(x,y)&=a_6(x,y)+a_8(y,x)\\ 2b_5(x,y)&=a_9(x,y)+a_{10}(y,x)\end{aligned}$$

and

$$b_i(y,x)=b_i'(x,y).$$

Let $\{x_n\}$ satisfy (v). Then from (i′)

$$\begin{aligned}M(Px_n,Px_m,t) &\geq b_1M(Qx_n,Px_n,\tfrac{t}{6})+b_1'M(Qx_m,Px_m,\tfrac{t}{6})\\ &+b_2M(Rx_n,Px_n,\tfrac{t}{6})+b_2'M(Rx_m,Px_m,\tfrac{t}{6})\\ &+b_3M(Qx_n,Px_m,\tfrac{t}{6})+b_3'M(Qx_m,Px_n,\tfrac{t}{6})\\ &+b_4M(Rx_n,Px_m,\tfrac{t}{6})+b_4'M(Rx_m,Px_n,\tfrac{t}{6})\\ &+b_5M(Qx_n,Rx_m,\tfrac{t}{6})+b_5'M(Qx_m,Rx_n,\tfrac{t}{6})\end{aligned}$$

$$\begin{aligned}&(1-b_3-b_3'-b_4-b_4'-b_5-b_5')M(Px_n,Px_m,t)\\ &\geq (b_1+b_3+b_5)M(Qx_n,Px_n,\tfrac{t}{2})+(b_1'+b_3'+b_5')M(Qx_m,Px_m,\tfrac{t}{2})\\ &+(b_2+b_4+b_5')M(Rx_n,Px_n,\tfrac{t}{2})+(b_2'+b_4'+b_5)M(Rx_m,Px_m,\tfrac{t}{2})\end{aligned}$$

From (ii) and (v), taking the limit as $m,n\to\infty$ shows that $\{Px_n\}$ is Cauchy, hence convergent call the limit z.

$$M(Qx_n,z,t)\geq M(Qx_n,Px_n,\tfrac{t}{2})+M(Px_n,z,\tfrac{t}{2})\to 1.$$

The continuity of Q and R implies

$$QPx_n\to Qz, Q^2x_n\to Qz, QRx_n\to Qz,$$
$$RPx_n\to Rz, R^2x_n\to Rz \text{ and } RQx_n\to Rz.$$

$$\begin{aligned}M(PRx_n,Rz,t) &\geq M(PRx_n,RPx_n,\tfrac{t}{2})+M(RPx_n,Rz,\tfrac{t}{2})\\ &\geq M(Rx_n,Px_n\tfrac{t}{2})+M(RPx_n,Rz,\tfrac{t}{2})\to 1\end{aligned}$$

So

$PRx_n\to Rz$, similarly $PQx_n\to Qz$

$$\begin{aligned}M(QRx_n,RQx_n,t) &\geq M(QRx_n,PQx_n,\tfrac{t}{3})\\ &+M(PQx_n,PRx_n,\tfrac{t}{3})+M(PRx_n,RQx_n,\tfrac{t}{3}).\end{aligned}$$

Using (i′)

$$\begin{aligned}M(PQx_n,PRx_n,t) &\geq b_1M(Q^2x_n,PQx_n,\tfrac{t}{10})\\ &+b_1'M(QRx_n,PRx_n,\tfrac{t}{10})+b_2M(RQx_n,PQx_n,\tfrac{t}{10})\\ &+b_2'M(R^2x_n,PRx_n,\tfrac{t}{10})+b_3M(Q^2x_n,PRx_n,\tfrac{t}{10})\\ &+b_3'M(QRx_n,PQx_n\tfrac{t}{10})+b_4M(RQx_n,PRx_n,\tfrac{t}{10})\\ &+b_4'M(R^2x_n,PQx_n,\tfrac{t}{10})+b_5M(Q^2x_n,R^2x_n,\tfrac{t}{10})\\ &+b_5'M(QRx_n,RQx_n,\tfrac{t}{10})\end{aligned}$$

$$\begin{aligned}&\geq b_1M(Q^2x_n,PQx_n,\tfrac{t}{10})+b_2'M(R^2x_n,PRx_n,\tfrac{t}{10})\\ &+b_3'M(QRx_n,PQx_n\tfrac{t}{10})+b_4M(RQx_n,PRx_n,\tfrac{t}{10})\\ &+(b_1'+b_2+b_3+b_4'+b_5+b_5')\max\{M(QRx_n,PRx_n,\tfrac{t}{10}),\\ &M(RQx_n,PQx_n,\tfrac{t}{10}),M(Q^2x_n,PRx_n,\tfrac{t}{10}),\\ &M(R^2x_n,PQx_n,\tfrac{t}{10}),M(Q^2x_n,R^2x_n,\tfrac{t}{10}),\\ &M(QRx_n,RQx_n,\tfrac{t}{10})\}.\end{aligned}$$

Hence

$$\limsup_n M(QRx_n, RQx_n, t)$$
$$\geq \limsup_n (b_1' + b_2 + b_3 + b_4' + b_5 + b_5')M(Qz, Rz, \frac{t}{10})$$

i.e

$$M(Qz, Rz, t) \geq \sup_{x,y\in X} (b_1' + b_2 + b_3 + b_4' + b_5 + b_5')M(Qz, Rz, \frac{t}{10})$$

which from (ii) implies that $Qz = Rz$.

From (i′)

$$M(PRx_n, Pz, t) \geq b_1 M(QRx_n, PRx_n, \frac{t}{10}) + b_1' M(Qz, Pz, \frac{t}{10})$$
$$+ b_2 M(R^2x_n, PRx_n, \frac{t}{10}) + b_2' M(Rz, Pz, \frac{t}{10})$$
$$+ b_3 M(QRx_n, Pz, \frac{t}{10}) + b_3' M(Qz, PRx_n, \frac{t}{10})$$
$$+ b_4 M(R^2x_n, Pz, \frac{t}{10}) + b_4' M(Rz, PRx_n, \frac{t}{10})$$
$$+ b_5 M(QRx_n, Rz, \frac{t}{10}) + b_5' M(Qz, R^2x_n, \frac{t}{10})$$
$$M(QPz, PQz, t) \geq M(Pz, Qz, t) = 1$$

$$\geq b_1 M(QRx_n, PRx_n, \frac{t}{10}) + b_2 M(R^2x_n, PRx_n, \frac{t}{10})$$
$$+ b_3' M(Qz, PRx_n, \frac{t}{10}) + b_4' M(Rz, PRx_n, \frac{t}{10})$$
$$+ b_5 M(QRx_n, Rz, \frac{t}{10}) + b_5' M(Qz, R^2x_n, \frac{t}{10})$$
$$+ (b_1' + b_2' + b_3 + b_4)\max\{M(Qz, Pz, \frac{t}{10}),$$
$$M(Rz, Pz, \frac{t}{10}), M(QRx_n, Pz, \frac{t}{10}),$$
$$M(R^2x_n, Pz, \frac{t}{10})\}.$$

Taking the limsup of both sides yield

$$M(Rz, Pz, t) \geq \limsup_n (b_1' + b_2' + b_3 + b_4)$$
$$\max\{M(Rz, Pz, \frac{t}{10}), M(Qz, Pz, \frac{t}{10})\}$$
$$\geq \sup_{x,y\in X} (b_1' + b_2' + b_3 + b_4)M(Rz, Pz, \frac{t}{10})$$

which from (ii), implies that $Rz = Pz$, since

$$M(Qz, Pz, t) \geq M(Qz, Rz, \frac{t}{2}) + M(Rz, Pz, \frac{t}{2}).$$

From (i′)

$$M(Pz, P^2z, t) \geq b_1 M(Qz, Pz, \frac{t}{10}) + b_1' M(QPz, P^2z, \frac{t}{10})$$
$$+ b_2 M(Rz, Pz, \frac{t}{10}) + b_2' M(RPz, P^2z, \frac{t}{10})$$
$$+ b_3 M(Qz, P^2z, \frac{t}{10}) + b_3' M(QPz, Pz, \frac{t}{10})$$
$$+ b_4 M(Rz, P^2z, \frac{t}{10}) + b_4' M(RPz, Pz, \frac{t}{10})$$
$$+ b_5 M(Qz, RPz, \frac{t}{10}) + b_5' M(QPz, Rz, \frac{t}{10})$$

$$M(QPz, P^2z, t) \geq M(QPz, PQz, \frac{t}{2})$$
$$+ M(PQz, P^2z, \frac{t}{2})$$

since $Qz = Pz$, $PQz = P^2z$. From (iv)

And $QPz = P^2z$. Similarly $RPz = P^2z$.

Therefore

$$M(Pz, P^2z, \frac{t}{10}) \geq b_3 M(Qz, Pz, \frac{t}{10}) + b_3 M(Pz, P^2z, \frac{t}{10})$$
$$+ b_3' M(QPz, P^2z, \frac{t}{10}) + b_3' M(P^2z, Pz, \frac{t}{10})$$
$$+ b_4 M(Rz, Pz, \frac{t}{10}) + b_4 M(Pz, P^2z, \frac{t}{10})$$
$$+ b_4' M(RPz, P^2z, \frac{t}{10}) + b_4' M(P^2z, Pz, \frac{t}{10})$$
$$+ b_5 M(Qz, Pz, \frac{t}{10}) + b_5 M(Pz, P^2z, \frac{t}{10})$$
$$+ b_5' M(P^2z, Pz, \frac{t}{10}) + b_5' M(Pz, Rz, \frac{t}{10}).$$

Thus

$$(1 - b_3 - b_3' - b_4 - b_4' - b_5 - b_5')M(Pz, P^2z, \frac{t}{10}) \leq 1$$

which from (ii) implies $Pz = P^2z$. Let $u = Pz$

$$M(Qu, u, t) = M(QPz, Pz, t)$$
$$\geq M(QPz, PQz, \frac{t}{2}) + M(PQz, Pz, \frac{t}{2})$$
$$\geq M(Pz, Qz, \frac{t}{2}) + M(P^2z, Pz, \frac{t}{2}) = 1$$

Similarly $Ru = u$.

Suppose u and v are common fixed points of P, Q and R. Then

$$M(u,v,t)=M(Pu,Pv,t)\geq b_3M(u,v,\frac{t}{6})+b_3'M(v,u,\frac{t}{6})$$
$$+b_4(u,v,\frac{t}{6})+b_4'M(v,u\frac{t}{6})+b_5M(u,v,\frac{t}{6})+b_5'M(v,u,\frac{t}{6})$$

which from (ii), implies $u=v$ and the fixed point is unique.

Let $\{y_n\}$ be any sequence in X with limit u and assume that

$$\sup_{x,y\in X}(b_1+b_2+b_3'+b_4')<1$$

$$M(Py_n,Pu,t)\geq b_1M(Qy_n,Py_n,\frac{t}{10})+b_1'M(Qu,Pu,\frac{t}{10})$$
$$+b_2M(Ry_n,Py_n,\frac{t}{10})+b_2'M(Ru,Pu,\frac{t}{10})$$
$$+b_3M(Qy_n,Pu,\frac{t}{10})+b_3'M(Qu,Py_n,\frac{t}{10})$$
$$+b_4M(Ry_n,Pu,\frac{t}{10})+b_4'M(Ru,Py_n,\frac{t}{10})$$
$$+b_5M(Qy_n,Ru,\frac{t}{10})+b_5'M(Qu,Ry_n,\frac{t}{10})$$
$$\geq b_1M(Qy_n,Pu,\frac{t}{10})+b_1M(Pu,Py_n,\frac{t}{10})$$
$$+b_2M(Ry_n,Pu,\frac{t}{10})+b_2M(Pu,Py_n,\frac{t}{10})$$
$$+b_3M(Qy_n,Pu,\frac{t}{10})+b_3'M(Qu,Pu,\frac{t}{10})$$
$$+b_3'M(Pu,Py_n,\frac{t}{10})+b_4M(Ry_n,Pu,\frac{t}{10})$$
$$+b_4'M(Ru,Pu,\frac{t}{10})+b_4'M(Pu,Py_n,\frac{t}{10})$$
$$+b_5M(Qy_n,Ru,\frac{t}{10})+b_5'M(Qu,Ry_n,\frac{t}{10})$$
$$\geq b_1M(Qy_n,Pu,\frac{t}{10})+b_2M(Ry_n,Pu,\frac{t}{10})$$
$$+b_3M(Qy_n,Pu,\frac{t}{10})+b_4M(Ry_n,Pu,\frac{t}{10})$$
$$+b_5M(Qy_n,Ru,\frac{t}{10})+b_5'M(Qu,Ry_n,\frac{t}{10})$$
$$+(b_1+b_2+b_3'+b_4')M(Pu,Py_n,\frac{t}{10})$$

taking the limsup on both sides yields

$$\limsup_n M(Py_n,Pu,t)$$
$$\geq \limsup_n(b_1+b_2+b_3'+b_4')\limsup_n M(Pu,Py_n,\frac{t}{10})$$
$$\geq \sup_{x,y\in X}(b_1+b_2+b_3'+b_4')\limsup_n M(Pu,Py_n,\frac{t}{10})$$

this from the assumption yields $\limsup_n M\left(Pu,Py_n,t\right)=1$. Therefore

$\limsup_n M\left(Pu,Py_n,\frac{t}{10}\right)=1$ and P is continuous at u.

Corollary Let X be a complete fuzzy metric space, A a self map of X satisfying

$$M(Ax,Ay,t)\geq c_1(x,y)M(x,Ax,\frac{t}{5})$$
$$\text{(vi)}\quad +c_1'(x,y)M(y,Ay,\frac{t}{5})+c_2(x,y)M(x,Ay,\frac{t}{5})$$
$$+c_2'(x,y)M(y,Ax,\frac{t}{5})+c_3(x,y)M(x,y,\frac{t}{5})$$

where c_i are non-negative and bounded and satisfy

$$\max\{\sup_{x,y\in X}(c_2+c_2'+c_3),\ \sup_{x,y\in X}(c_1'+c_2)\}<1$$

if there exists an asymptotically regular sequence X then A has a unique fixed point.

Example Let (X,d) be metric space. Define $a*b=\min\{a,b\}$(or $a*b=ab$) and $M(x,y,t)=\frac{t}{t+|x-y|}$ for all $x,y\in X$ and all $t>0$. Then $(X,M,*)$ is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d, the standard fuzzy metric.

Example Let $X=[0,2]$ and $a*b=\min\{a,b\}$. Let M be the standard fuzzy metric induced by d. Define three self maps P,Q and R of a fuzzy metric space $(X,M,*)$ as follows

$$Px=\begin{cases}1-x, & 0\leq x\leq 1\\ 2\ , & 1\leq x\leq 2\end{cases}$$
$$Qx=\begin{cases}\frac{x}{2}\ , & 0\leq x\leq 1\\ 2\ , & 1\leq x\leq 2\end{cases}$$
$$Rx=\begin{cases}\frac{x}{4}\ , & 0\leq x\leq 1\\ 2\ , & 1\leq x\leq 2\end{cases}$$

Taking $x_n=1-\frac{1}{n}$ then $x_n\to 1$ as $n\to\infty$ also $Px_n,Qx_n,Rx_n\to 1$ as $n\to\infty$.

The mappings P,Q and R is define the set $[1,2]$. For any $x\in[1,2]$.

$$PQx=2=QPx$$
$$PRx=2=RPx$$

Thus $\{P,Q\}$ and $\{P,R\}$ are weakly commuting.

Theorem 3.2 Let P,Q be self- maps of a complete fuzzy metric space X satisfying

(i) $$M(Px,Py,t)\geq a_1M(Qx,Px,\frac{t}{5})+a_2M(Qy,Py,\frac{t}{5})$$
$$+a_3M(Qx,Py,\frac{t}{5})+a_4M(Qy,Px,\frac{t}{5})+a_5M(Qx,Qy,\frac{t}{5})$$
for all $x,y\in X$. Where the $a_i=a_i(x,y)\geq 0 \quad (i=1,\cdots,5)$

(ii) $\{P,Q\}$ is a weakly commuting pair

(iii) There exists a sequence $\{x_n\}$ in X which is asymptotically regular with respect to $\{P,Q\}$.

(iv) P and Q are continuous.

Then P and Q have a unique common fixed point.

Prove can be given as above.

Result in Compact Fuzzy Metric Space

In this section we are extended to generalize non-expansive mapping defined over a compact fuzzy metric space.

Theorem 3.3 Let X be a compact fuzzy metric space, T a self map of X satisfying (vii) with $\sup_{x,y\in X}(c_1+c_1'+c_2+c_2'+c_3)(x,y)\leq 1$ and having property (M) over X. Suppose also that, $Y\subset X$ such that $Ty\subset X, x_n\to x$ implies $Tx_n\to x,\ x_n\in X$. Then T has a unique fixed point in X, provided $c_2=c_2'$ and $\sup_{x,y\in X}(c_1'+c_2)(x,y)<1$.

Proof Let $CL(X)$ denote the collection of subsets $K\subset X$ which are non-empty, closed and invariant under T.

If K is singleton, and then it is a fixed point of T. If K contains more than one point, then by property *(M)*, there is an $x\in K$ such that

$$(*)\qquad M(x,Tx,t)=1-r\geq \sup_{y\in K}M(y,Ty,t)$$

Let $K_1=\{x\in K: M(x,Tx,t)\geq 1-r\}$. Then by (*) K_1 is a non-empty proper subset of $n\to\infty$. Also for $x\in K_1$, since T satisfies (vii) we have

$$M(Tx,T^2x,t)\geq c_1M(x,Tx,\frac{t}{2})+c_1'M(Tx,T^2x,\frac{t}{2})$$
$$+c_2M(x,T^2x,\frac{t}{2})+c_2'M(Tx,Tx,\frac{t}{2})+c_3M(x,Tx,\frac{t}{2})$$
$$(1-c_2'-c_2)M(Tx,T^2x,t)\geq(c_1+c_2+c_3)M(x,Tx,\frac{t}{2})$$
$$M(Tx,T^2x,t)\geq\frac{(c_1+c_2+c_3)}{(1-c_1'-c_2)}M(x,Tx,\frac{t}{2})\geq M(x,Tx,\frac{t}{2})$$

and K_1 is invariant under T.

Let x be a limit point of K_1. Then there exists a sequence $\{x_n\}\subset K_1$ with $x_n\to x$. By hypothesis $Tx_n\to x$

$$M(x,Tx,t)\geq M(x,Tx_n,\frac{t}{2})+M(Tx_n,Tx,\frac{t}{2})$$
$$\geq M(x,Tx_n,\frac{t}{2})+c_1M(x_n,Tx_n,\frac{t}{2})+c_1'M(x,Tx,\frac{t}{2})$$
$$+c_2M(x_n,Tx,\frac{t}{2})+c_2'M(x,Tx_n,\frac{t}{2})+c_3M(x_n,x,\frac{t}{2})$$
$$\geq M(x,Tx_n,\frac{t}{2})+c_1(1-r)+c_1'M(x,Tx,\frac{t}{2})$$
$$+c_2M(x_n,x,\frac{t}{2})+c_2M(x,Tx,\frac{t}{2})+c_2'M(x,Tx_n,\frac{t}{2})$$
$$+c_3M(x_n,x,\frac{t}{2})$$

therefore

$$M(x,Tx,t)\geq\frac{1}{(1-c_1'-c_2)}[M(x,Tx_n,\frac{t}{2})+c_2M(x_n,x,\frac{t}{2})$$
$$+c_2'M(x,Tx_n,\frac{t}{2})+c_3M(x_n,x,\frac{t}{2})]+\frac{c_1(1-r)}{(1-c_1'-c_2)}$$
$$\geq\frac{(1+c_2+c_2'+c_3)}{(1-c_1'-c_2)}\max\{M(x,Tx_n,\frac{t}{2}),M(x_n,x,\frac{t}{2})\}+(1-r)$$
$$\geq\frac{2}{1-\sup_{x,y\in X}(c_1'+c_2)}\max\{M(x,Tx_n,\frac{t}{2}),M(x_n,x,\frac{t}{2})\}+(1-r)$$

Taking the limit as $n\to\infty$ yields $M(x,Tx,t)\geq 1-r$ and K_1 is closed, so K_1 contain only one point, which is the unique fixed point of T.

Remark (1) In Theorem (3.3) T is assumed continuous, then the condition $x_n\to x$ implies $Tx_n\to x$ for $x_n\in Y$.

Theorem 3.4 Let X be a compact fuzzy metric space, T a continuous self map of x satisfying (vii) with $\sup_{x,y\in X}(2c_1'+c_2+c_2'+c_3)(x,y)\leq 1,\ c_2=c_2'$ and $\sup_{x,y\in X}(c_1'+c_2)(x,y)<1$. Suppose that T satisfies property (M) over X and that $M(Tx,p,t)>M(x,p,t)$ for each $x\neq p, p$ the unique fixed point of T. Then for each $x\in X,\ T^nx\to p.$

Proof From the remark (1), T has a unique fixed point p. Let $x\in X$ then from compactness of X, $\{T^{n_i}x\}$ contains a convergent subsequence $\{T^{n_i}x\}$. Let $z=\lim_i T^{n_i}x$. Using (vii)

$$M(p,T^n x,t) \geq c_1 M(p,Tp,\frac{t}{2}) + c_1' M(T^{n-1}x,T^n x,\frac{t}{2})$$
$$+ c_2 M(p,T^n x,\frac{t}{2}) + c_2' M(T^{n-1}x,Tp,\frac{t}{2})$$
$$+ c_3 M(p,T^{n-1}x,\frac{t}{2})$$
$$\Rightarrow \quad M(p,T^n x,t) \geq \frac{(c_1' + c_2' + c_3)}{(1-c_1'-c_2)} M(p,T^{n-1}x,\frac{t}{2})$$
$$\geq M(p,T^{n-1}x,\frac{t}{2})$$

Therefore $\{M(p,T^n x,t)\}$ is non-negative increasing sequence and hence converges. This, along with the convergence of $\{T^{n_i}x\}$ implies $M(T^n x,p,t) \to M(z,p,t)$ if $z \neq p$ then the hypothesis of the theorem is contradicted.

4. Conclusions

Hardy and Rogers proved in [5] a fixed point theorem for complete metric space and to establish non-expansive mapping defined over a compact metric space. In the present paper, we have generalized Hardy-Rogers type mappings on complete fuzzy metric space and also extend to generalize non-expansive mapping defined over compact fuzzy metric space.

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