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Two Weight Characterization of New Maximal Operators

Hu Yunpeng, Cao Yonghui

Department of Mathematics and Systems Sciences, Xinjiang University, Urumqi, China

Email address:
husousuke@163.com (Hu Yunpeng),13579402215@163.com (Cao Yonghui)

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Abstract: For the last twenty years, there has been a great deal of interest in the theory of two weight. In the present paper, we investigate the two weight norm inequalities for fractional new maximal operator on the Lebesgue space. More specifically, we obtain that the sufficient and necessary conditions for strong and weak type two weight norm inequalities for a new fractional maximal operators by introducing a class of new two weight functions. In the discussion of strong type two weight norm inequalities, we make full use of the properties of dyadic cubes and truncation operators, and utilize the space decomposition technique which space is decomposed into disjoint unions. In contrast, weak type two weight norm inequalities are more complex. We have the aid of some good properties of A_p weight functions and ingeniously use the characteristic function. What should be stressed is that the new two weight functions we introduced contains the classical two weights and our results generalize known results before. In this paper, it is worth noting that $w(x)dx$ may not be a doubling measure if our new weight functions $\omega \in A_p(\varphi)$.. Since $\varphi(|Q|) \geq 1$, our new weight functions are including the classical Muckenhoupt weights.

Keywords: Two Weight, Maximal Operator, Lebesgue Space

1. Introduction

In 1972, Muckenhopt [1] established the A_p weight theory when studying the Lebesgue boundedness of classical Hardy-Littlewood maximal operators. Subsequently, he further obtained the two weight boundedness of Hardy-Littlewood maximal operators and founded that the A_p two weight condition is a necessary and sufficient condition for the two weight weak boundedness of Hardy-Littlewood maximal operators. In addition, he discussed strong boundedness. Muckenhopt and Wheeden [2] founded that the A_p two weight condition is only a necessary but not sufficient condition for Hardy-Littlewood maximal operator and Hilbert transform to have two weight strong boundedness, which is essentially different from the one weight case. Therefore, as a generalization of the one weight case, it is more difficult to discuss the boundedness of operators with two weights than with one weight. In 1982, great progress was made in the two weighted results. Sawyer [3] obtained the necessary and sufficient conditions for the two weight (u, v) of Hardy-Littlewood maximal operators to be bounded from $L^p(vdx)$ to $L^p(udx)$ with $1 < p \leq \infty$. In 2000, Cruz-Uribe [4] gave a new proof of this result, and we can see a lot of work related to this topic in this paper, for example, see [5-8].

Next, some necessary definitions and notations are given. In this paper, $Q(x,r)$ denotes the cube centered at x and of the sidelength t. Similarly, given $Q = Q(x,r)$ and $\lambda > 0$, we will write λQ for the λ-dilate cube, which is the cube with the same center x and with sidelength λt. C denote constants independent of parameters and may take different values in different places. Let $E \in R^n$, χ_E denote the characteristic function of E. $|E|$ denote the Lebesgue measure of E. p' denote the conjugate index of p, i.e. $1/p + 1/p' = 1$.

Definition 1.1[9]. Let $\varphi(t) = (1+t)^{\alpha_0}, t \geq 0, \alpha_0 \geq 0$. $\omega \geq 0\ a.e.$ and $\omega \in L^p_{loc}(R^n)$.

(i) If there is a constant C such that for all cubes $Q = Q(x,r)$ with radius r centered on x,

$$\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega(y)dy\right)\left(\frac{1}{\varphi(|Q|)|Q|}\int_Q \omega^{-\frac{1}{p-1}}(y)dy\right)^{p-1} \leq C,$$

Then we say that $\omega \in A_p(\varphi)(p > 1)$.

(ii) If there is a constant C such that

$$M_\varphi(\omega)(x) \leq C\omega(x), a.e. x \in R^n,$$

where,

$$M_{\varphi}f(x)=\sup_{x\in Q}\frac{1}{\varphi(|Q|)|Q|}\int_{Q}|f(y)|dy,$$

Then we say that $\omega\in A_1(\varphi)$.

The classical Hardy-Littlewood maximal operator M is defined by

$$Mf(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_{Q}|f(y)|dy.$$

Obviously, $f(x)\le M_{\varphi}f(x)\le Mf(x), a.e. x\in R^n$ and $M_{\varphi}f(x)$ is lower semi-continuous.

Since $\varphi(|Q|)\ge 1$, so $A_p(R^n)\subset A_p(\varphi)$ for $1<p<\infty$, where $A_p(R^n)$ denote the classical Muckenhoupt weights; see [10]. It is well know that if $\omega\in A_\infty(R^n)=\bigcup_{p\ge 1}A_p(R^n)$, then $\omega(x)dx$ be a doubling measure, i.e. there exist a constant $C>0$ for any cube Q such that

$$\omega(2Q)\le C\omega(Q).$$

But Tang pointed out in [9] that if $\omega\in A_p(\varphi)$, then $\omega(x)dx$ may not be a doubling measure. In fact, let $0\le\gamma\le n\alpha_0$, It is easy to check that $\omega(x)=(1+|x|)^{-(n+\gamma)}\notin A_\infty(R^n)$ and $\omega(x)dx$ are not a double measures, but $\omega(x)=(1+|x|)^{-(n+\gamma)}\in A_1(\varphi)$.

The new maximal operators were firstly introduced by Tang [9] in connection with weighted L^p inequalities for pseudo-difffferential operators with smooth symbols and their commutators by using a class of new weight functions which include Muckenhoupt weight functions. It can control a class of important operators, such as pseudo-difffferential operator

$$Tf(x):=\int_{R^n}\sigma(x,\xi)e^{2\pi ix\cdot\xi}\hat{f}(\xi)d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f, Symbol $\sigma(x,\xi)\in S^0_{1,\delta}$ with $0<\delta\le 1$. In particular, A. Laptev [11] proved that any $S^0_{1,0}$ pseudo-difffferential operator is a standard Calderón-Zygmund operator; see also [12 − 15]. Applying the new maximal functions, Pan [16] obtained the strong type and weak end-point estimates for certain classes of multilinear operators and their iterated commutators with new BMO functions.

In this paper, the fractional form of a new maximal operator considered by Tang in [9] is introduced: given $0\le\alpha<1$, the fractional new maximal function $M^{\alpha}_{\varphi}f(x)$ is defined as

$$M^{\alpha}_{\varphi}f(x):=\sup_{x\in Q}\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_{Q}|f(y)|dy,$$

where the supremum is taken over all the cube containing x.

Next, we introduce a new class of two weight functions, which includes the classical two weights in [3].

Definition 1.2. Let $1\le p\le q<\infty, 0\le\alpha<1$ and $u,v\in L^1_{loc}(R^n)$. If for every cube Q,

(i) When $1<p<\infty$,

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_{Q}u(x)dx\right)^{\frac{1}{q}}\left(\int_{Q}v(x)^{1-p'}dx\right)^{\frac{1}{p'}}\le C.$$

(ii) When $p=1$, for a.e. $x\in Q$,

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_{Q}u(x)dx\right)^{\frac{1}{q}}\le Cv(x).$$

Then we say that $(u,v)\in A^{\alpha}_{p,q}(\varphi)$. It is easy to know that $A^{\alpha}_{p,q}(R^n)\subset A^{\alpha}_{p,q}(\varphi)$, with $A^{\alpha}_{p,q}(R^n)$ denote classical two weight.

We end this section with the outline of this paper. Section 2 contains Theorem 2.1.1, Theorem 2.2.1 and the proofs of them. we extend the corresponding strong and weak results to the fractional new maximal operators. In Section 3, we give a conclusion.

2. Method and Results

2.1. Strong Boundedness of New Maximal Operators

The purpose of this paper is to study the strong and weak type inequalities of fractional new maximal operators by introducing a new class of two weight functions containing the classical two weight, and obtain their two weight characterization in Lebesgue spaces.

The main result of this section is to obtain the two weight strong boundedness of fractional new maximal operators.

Theorem 2.1.1. Let $1<p\le q<\infty, 0\le\alpha<1$ and $(u,v)\in A^{\alpha}_{p,q}(\varphi)$. Then the following statements are equivalent:

(i) For every cube Q,

$$\left(\int_{Q}\left(M^{\alpha}_{\varphi}(v^{1-p'}\chi_Q)(x)\right)^{q}u(x)dx\right)^{1/q}\le C\left(\int_{Q}v(x)^{1-p'}dx\right)^{1/p};$$

(ii) For every $f\in L^p(v)$,

$$\left(\int_{R^n}\left(M^{\alpha}_{\varphi}f(x)\right)^{q}u(x)dx\right)^{1/q}\le C\left(\int_{R^n}|f(x)|^p v(x)dx\right)^{1/p}.$$

The following lemma is needed to prove the theorem.

Lemma 2.1.2. Let $0\le\alpha<1, f\ge 0$ is a locally integrable function. If for every cube Q and some $t>0$ such that

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_{Q}|f(y)|dy>t,$$

Then there exists a dyadic cube P such that $Q\subset 3P$ and

$$\frac{1}{(\varphi(|P|)|P|)^{1-\alpha}}\int_P f(y)dy > 2^{\alpha-1-n}t.$$

Proof. Take $k \in Z$ makes $2^{k-1} \le \ell(Q) < 2^k$, so there exists dyadic cubes $P_1, P_2, \dots, P_N, 1 \le N \le 2^n$ whose generate by 2^k and intersect Q. Since $\ell(P_j) = 2^k > l(Q)$, by Vitali covering lemma, we get $Q \subset 3P_j$ for every j. In addition, it is assumed that there is no dyadic cube P such that

$$\int_P f(y)dy > \frac{t(\varphi(|Q|)|Q|)^{1-\alpha}}{2^n}.$$

Then,

$$\int_Q f(y)dy \le \sum_{j=1}^{N}\int_{P_j} f(y)dy \le \sum_{j=1}^{N}\frac{t(\varphi(|Q|)|Q|)^{1-\alpha}}{2^n} \le t(\varphi(|Q|)|Q|)^{1-\alpha}.$$

This contradicts the hypothesis. Therefore, there exists a dyadic cube P,

$$\frac{1}{(\varphi(|P|)|P|)^{1-\alpha}}\int_P f(y)dy > \frac{t(\varphi(|Q|)|Q|)^{1-\alpha}}{2^n(\varphi(|P|)|P|)^{1-\alpha}} \ge 2^{\alpha-1-n}t.$$

Next, we give the proof of the theorem.

Proof of Theorem 2.1.1. Set $\sigma(x) = v(x)^{1-p'}$. In (ii), take $f = \sigma\chi_Q$, it is easy to see that (ii) implicature (i). Next, we prove that (i) implicature (ii). Firstly, without losing generality, Let $f \in L^p(v)$ be a nonnegative bounded function with compact support. This ensures that $M_\varphi^\alpha f(x)$ is almost everywhere finite. R^n is decomposed in the following way:

$$R^n = \bigcup_{k\in Z}\Omega_k, \Omega_k = \{x \in R^n: 2^k < M_\varphi^\alpha f(x) \le 2^{k+1}\}.$$

Then, for each k and $x \in \Omega_k$, there exists a cube Q_x^k containing x such that

$$\frac{1}{(\varphi(|Q_x^k|)|Q_x^k|)^{1-\alpha}}\int_{Q_x^k} f(y)dy > 2^k,$$

Therefore, by Lemma 2.1.2, there exists a dyadic cube P_x^k such that $Q_x^k \subset 3P_x^k$ and

$$\frac{1}{(\varphi(|P_x^k|)|P_x^k|)^{1-\alpha}}\int_{P_x^k} f(y)dy > 2^{\alpha-1-n}2^k. \qquad (1)$$

This estimate shows that for each fixed k, the dyadic cube P_x^k is bounded. Therefore, there exists such a succollection of disjoint dyadic cubes $\{P_j^k\}_j$ that for a j, each Q_x^k is contained in $3P_j^k$. Therefore, $\Omega_k \subset \bigcup_j 3P_j^k$. Next, we decompose the Ω_k:

$$E_1^k = 3P_1^k \cap \Omega_k, E_2^k = (3P_2^k \backslash 3P_1^k) \cap \Omega_k, \dots, E_j^k = \left(3P_j^k \backslash \bigcup_{r=1}^{j-1} 3P_r^k\right) \cap \Omega_k, \dots$$

So we get

$$R^n = \bigcup_{k=Z}\Omega_k = \bigcup_{j,k} E_j^k,$$

where E_j^k's are pairwise disjont for all j and k. Fix a sufficiently large constant $K > 0$ and Set $\Lambda_K = \{(j,k) \in \mathbb{N} \quad \mathbb{Z}: |k| \le K\}$. Since P_j^k satisfies (1), using $E_j^k \subset \Omega_k$, we get

$$I_k = \int_{\bigcup_{k=-K}^{K}\Omega_k}\left(M_\varphi^\alpha f(x)\right)^q u(x)dx = \sum_{(j,k)\in\Lambda_K}\int_{E_j^k}\left(M_\varphi^\alpha f(x)\right)^q u(x)dx$$

$$\le \sum_{(j,k)\in\Lambda_K} u(E_j^k)(2^{k+1})^q \le C\sum_{(j,k)\in\Lambda_K} u(E_j^k)\left(\frac{1}{\left(\varphi(|P_j^k|)|P_j^k|\right)^{1-\alpha}}\int_{P_j^k} f(y)dy\right)^q$$

$$= C\sum_{(j,k)\in\Lambda_K} u(E_j^k)\left(\frac{1}{\left(\varphi(|3P_j^k|)|3P_j^k|\right)^{1-\alpha}}\int_{3P_j^k}\sigma(y)dy\right)^q\left(\frac{\int_{P_j^k}(f\sigma^{-1})(y)\sigma(y)dy}{\int_{3P_j^k}\sigma(y)dy}\right)^q$$

$$= C\int_{\mathcal{Y}} T_k(f\sigma^{-1})^q dv, \qquad (2)$$

where $\mathcal{Y} = \mathbb{N} \quad \mathbb{Z}$, v is a measure in $\mathcal{Y}$, with

$$v(j,k) = u(E_j^k)\left(\frac{1}{\left(\varphi(|3P_j^k|)|3P_j^k|\right)^{1-\alpha}}\int_{3P_j^k}\sigma(y)dy\right)^q,$$

For each measurable function h, operator T_k is defined by

$$T_k h(j,k) = \frac{\int_{P_j^k} h(y)\sigma(y)dy}{\int_{3P_j^k}\sigma(y)dy}\chi_{\Lambda_K(j,k)}.$$

In this case, if we can be proved that $T_k: L^p(R^n,\sigma) \to$

$L^q(\mathcal{Y},v)$ is uniformly bounded, then it can be obtained from (2),

$$I_k \le C\int_{\mathcal{Y}} T_k(f\sigma^{-1})^q dv \le C\left(\int_{R^n}(f\sigma^{-1})^p\sigma dx\right)^{\frac{q}{p}} = C\left(\int_{R^n} f^p v dx\right)^{\frac{q}{p}}.$$

The expected inequalities can be obtained by using the consistency of I_k and the control convergence theorem. Therefore, only proving that $T_k: L^p(R^n,\sigma)\to L^q(\mathcal{Y},v)$ uniformly bounded is necessary. $T_k: L^\infty(R^n,\sigma)\to L^\infty(\mathcal{Y},v)$ is obvious, by Marcinewicz interpolation theorem, it is only to prove the uniform boundedness of $T_k: L^1(R^n,\sigma)\to L^{q/p,\infty}(\mathcal{Y},v)$. So we only need to prove that

$$v\{(j,k)\in\mathcal{Y}: T_k h(j,k)>\lambda\} \le C\left(\frac{1}{\lambda}\int_{R^n}|h(x)|\,\sigma(x)dx\right)^{q/p}, \lambda>0.$$

For this reason, fix $h\ge 0$ is a bounded function with compact support. Set

$$F_\lambda = \{(j,k)\in\mathcal{Y}: T_k h(j,k)>\lambda\} = \{(j,k)\in\Lambda_K: T_k h(j,k)>\lambda\}.$$

Since $E_j^k\subset 3P_j^k$, we have

$$v(F_\lambda) = \sum_{(j,k)\in F_\lambda} u(E_j^k)\left(\frac{1}{\left(\varphi(|3P_j^k|)|3P_j^k|\right)^{1-\alpha}}\int_{3P_j^k}\sigma(y)dy\right)^q$$

$$\le \sum_{(j,k)\in F_\lambda}\int_{E_j^k}\left(\frac{1}{\left(\varphi(|3P_j^k|)|3P_j^k|\right)^{1-\alpha}}\int_{3P_j^k}\sigma(y)dy\right)^q u(x)dx$$

$$\le \sum_{(j,k)\in F_\lambda}\int_{E_j^k}\left(M_\varphi^\alpha\left(\sigma\chi_{3P_j^k}\right)(x)\right)^q u(x)dx.$$

The dydic cube in the family $\{P_j^k:(j,k)\in F_\lambda\}$ is bounded. In fact, if $(j,k)\in F_\lambda$, then $|k|\le K$, and for each k, the cube $\{P_j^k\}_j$ is bounded. This allows us to select the largest subset family $\{P_i\}_i$ for each $(j,k)\in F_\lambda, P_j^k\subset P_i$. By E_j^k's are pairwise disjoint and $E_j^k\subset 3P_j^k$,

$$v(F_\lambda)\le\sum_i\sum_{P_j^k\subset P_i}\int_{E_j^k}\left(M_\varphi^\alpha\left(\sigma\chi_{3P_j^k}\right)(x)\right)^q u(x)dx$$

$$\le\sum_i\int_{3P_j^k}\left(M_\varphi^\alpha\left(\sigma\chi_{3P_j^k}\right)(x)\right)^q u(x)dx$$

$$\le C\sum_i\left(\int_{3P_i}\sigma(x)dx\right)^{q/p},$$

where we used the definition of two weight. Since the cube P_i is selected from the maximal disjoint subcollection $\{P_j^k:(j,k)\in F_\lambda\}$, there exists $P_i=P_j^k$ for each i, with $(j,k)\in F_\lambda$. In this case, $T_k h(j,k)>\lambda$, and since $(j,k)\in F_\lambda$,

$$\int_{3P_i}\sigma(x)dx = \int_{3P_j^k}\sigma(x)dx < \frac{1}{\lambda}\int_{P_j^k}h(x)\sigma(x)dx = \frac{1}{\lambda}\int_{P_i}h(x)\sigma(x)dx.$$

Notice that $q/p\ge 1$, the maxima of cube P_i and their disjoints, we have

$$v(F_\lambda)\le C\sum_i\left(\frac{1}{\lambda}\int_{P_i}h(x)\sigma(x)dx\right)^{q/p}$$

$$\le C\left(\sum_i\frac{1}{\lambda}\int_{P_i}h(x)\sigma(x)dx\right)^{q/p}$$

$$\le C\left(\frac{1}{\lambda}\int_{R^n}h(x)\sigma(x)dx\right)^{q/p},$$

where C is independent of Q, so the proof is completed.

2.2. Weak Boundedness of New Maximal Operators

The main result of this section is to obtain the two weight weak boundedness of fractional new maximal operators.

Theorem 2.2.1. Let $1\le p\le q<\infty, 0\le\alpha<1$ and $(u,v)\in A_{p,q}^\alpha(\varphi)$. Then the following statements are equivalent:

(i) $(u,v)\in A_{p,q}^\alpha(\varphi)$;

(ii) $M_\varphi^\alpha: L^p(v)\to L^{q,\infty}(u)$, i.e. for any $\lambda>0$,

$$u\{x\in R^n: M_\varphi^\alpha f(x)>\lambda\}\le\frac{C}{\lambda^q}\left(\int_{R^n}|f(x)|^p\,v(x)dx\right)^{q/p};$$

(iii) For every $f \geq 0$ and cube Q,

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx\right)^q u(Q) \leq C\left(\int_Q f(x)^p v(x)dx\right)^{q/p}.$$

Proof of Theorem 2.2.1. The proof will be carried out in the following manner:

(ii) ⇒ (iii) ⇒ (i) ⇒ (iii) ⇒ (ii)

(ii) ⇒ (iii) Take $f \geq 0$ and cube Q such that

$$\tilde{f}_{\alpha,Q} = \frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx > 0.$$

If $0 < \lambda < \tilde{f}_{\alpha,Q}$ and $x \in Q$, we have

$$\lambda < \tilde{f}_{\alpha,Q} = \frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)\chi_Q dx \leq M_\varphi^\alpha(f\chi_Q)(x),$$

Then $Q \subset \{y \in R^n : M_\varphi^\alpha(f\chi_Q)(y) > \lambda\}$. By (ii),

$$u(Q) \leq u\{y \in R^n : M_\varphi^\alpha(f\chi_Q)(y) > \lambda\} \leq \frac{C}{\lambda^q}\left(\int_Q f(x)^p v(x)dx\right)^{q/p},$$

Thus,

$$\left(\tilde{f}_{\alpha,Q}\right)^q u(Q) \leq C\left(\int_Q f(x)^p v(x)dx\right)^{q/p},$$

Therefore, (iii) hold.

(iii) ⇒ (i) For $p > 1$, Let $f \geq 0$. For any $S \subset Q$, in (iii), take $f = f\chi_S$ to have

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_S f(x)dx\right)^q u(Q) \leq C\left(\int_S f(x)^p v(x)dx\right)^{q/p}. \quad (3)$$

If $f \equiv 1$, then

$$\left(\frac{|S|}{(\varphi(|Q|)|Q|)^{1-\alpha}}\right)^q u(Q) \leq Cv(S)^{q/p}. \quad (4)$$

According to [5,388 pages], we can only consider non-trivial cases and prove that $(u,v) \in A_{p,q}^\alpha(\varphi)$. Firstly, for $1 < p < \infty$, take $f(x) = v(x)^{1-p'}$. Fix Q, Set

$$S_j = \left\{x \in Q : v(x) > \frac{1}{j}\right\}, j = 1,2, \ldots$$

Then f is bounded on each S_j and $\int_{S_j} v^{1-p'}dx < \infty$. Using (3), with $S = S_j$ and $f = f\chi_S$, we get

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_{S_j} v(x)^{1-p'}dx\right)^q u(Q) \leq C\left(\int_{S_j} v(x)^{1-p'}dx\right)^{q/p}.$$

Each integral is finite, so

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_{S_j} v(x)^{1-p'}dx\right)^{q-\frac{q}{p}} u(Q) \leq C,$$

i.e.

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(x)dx\right)^{\frac{1}{q}}\left(\int_{S_j} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} \leq C.$$

In addition, $S_1 \subset S_2 \subset \cdots$ and $\bigcup_j S_j = \{x \in Q : v(x) > 0\}$. Let $j \to \infty$, we get

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(x)dx\right)^{\frac{1}{q}}\left(\int_{\{x\in Q:v(x)>0\}} v(x)^{1-p'}dx\right)^{\frac{1}{p'}} \leq C.$$

Therefore, $v > 0$ $a.e.$, $(u,v) \in A_{p,q}^\alpha(\varphi)$.

For $p = 1$, notice that (4) can be written in the following form: for every Q, with $S \subset Q, |S| > 0$,

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(x)dx\right)^{1/q} \leq C\frac{v(S)}{|S|}.$$

Fix Q, consider

$$a > \operatorname*{ess\,inf}_Q v = \inf\{t > 0 : |\{x \in Q : v(x) < t\}| > 0\}.$$

Set $S_a = \{x \in Q : v(x) < a\} \subset Q$. Then, $|S_a| > 0$ and

$$\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(x)dx\right)^{1/q} \leq \frac{C}{|S_a|}\int_{S_a} v(x)dx \leq \frac{C}{|S_a|}a|S_a| = Ca.$$

For each $a > ess\,inf_Q\, v$, the upper formula is hold, so

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q u(x)dx\right)^{\frac{1}{q}} \leq C\operatorname*{ess\,inf}_Q v \leq Cv(x).$$

i.e. $(u,v) \in A_{1,q}^\alpha(\varphi)$.

(i) ⇒ (iii) First, we consider the case of $p = 1$. For $f \geq 0$ and every cube Q,

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx\right)^q u(Q)=\left(\int_Q f(x)\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(y)dy\right)^{\frac{1}{q}}dx\right)^q$$
$$\le C\left(\int_Q f(x)v(x)dx\right)^q.$$

The last inequality used $(u,v)\in A_{1,q}^{\alpha}(\varphi)$.
On the other hand, when $1<p<\infty$, the Hölder inequality has

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx\right)^q=\frac{1}{(\varphi(|Q|)|Q|)^{(1-\alpha)q}}\left(\int_Q f(x)v(x)^{\frac{1}{p}}v(x)^{-\frac{1}{p}}dx\right)^q$$
$$\le\frac{1}{(\varphi(|Q|)|Q|)^{(1-\alpha)q}}\left(\int_Q f(x)^p v(x)\,dx\right)^{q/p}\left(\int_Q v(x)^{1-p'}dx\right)^{q/p'}.$$

Since $(u,v)\in A_{p,q}^{\alpha}(\varphi)$,

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx\right)^q u(Q)$$
$$\le\frac{1}{(\varphi(|Q|)|Q|)^{(1-\alpha)q}}\left(\int_Q f(x)^p v(x)\,dx\right)^{q/p}\left(\int_Q v(x)^{1-p'}dx\right)^{q/p'}\int_Q u(x)dx$$
$$=\left(\int_Q f(x)^p v(x)\,dx\right)^{q/p}\left\{\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\left(\int_Q u(x)\,dx\right)^{1/q}\left(\int_Q v(x)^{1-p'}dx\right)^{1/p'}\right\}^q$$
$$\le C\left(\int_Q f(x)^p v(x)\,dx\right)^{q/p}.$$

Therefore, (iii) is hold.
(iii) ⇒ (ii) Let $f\in L_{loc}^p(R^n), u(Q)>0$, then

$$\left(\frac{1}{(\varphi(|Q|)|Q|)^{1-\alpha}}\int_Q f(x)dx\right)^q u(Q)$$
$$\le C\left(\int_Q f(x)^p v(x)\,dx\right)^{q/p}<\infty,$$

thus $f\in L_{loc}^1(R^n)$. So it can be assumed that $f\in L^1(R^n)$, by defining $f_k=f\chi_{Q(0,k)}$, then we have $f_k\to f$. In this case, the k limit of f_k in (ii) is independent of each constant C, so (ii) about f holds. With these, we will prove that the inequality of expectation holds when $f\ge 0, f\in L^p(v)\cap L^1(R^n)$.

Set

$$E_\lambda=\{x\in R^n: M_\varphi^\alpha f(x)>\lambda\}.$$

If $x\in E_\lambda$, from the definition of φ and the relationship between maximal function and central maximal function, there exists $r_x>0$ such that

$$\frac{1}{(\varphi(|r_x|^n)|r_x|^n)^{1-\alpha}}\int_{Q(x,r_x)} f(y)dy$$
$$>2^{\alpha-1}\left(\frac{1+2^n|Q|}{1+|Q|}\right)^{-\alpha_0}\lambda.$$

In particular, $|r_x|^n\le\varphi(|r_x|^n)^{-1}\left(2^{1-\alpha}\left(\frac{1+2^n|Q|}{1+|Q|}\right)^{\alpha_0}\lambda^{-1}||f||_{L^1}\right)^{\frac{1}{1-\alpha}}\le C$. By Vitali covering lemma, there exists a subcollection of pairwise disjoint cubes $\{Q(x_j,r_j)\}_j$, with $x_j\in E_\lambda, r_j=r_{x_j}$,

$$E_\lambda\subset\bigcup_{x\in E_\lambda}Q(x,r_x)\subset\bigcup_j Q(x,3r_j).$$

Recall that (iii) led to (3). Let $Q=Q(x_j,3r_j)$ and $S=Q(x_j,r_j)\subset Q$, then

$$u(E_\lambda)\le\sum_j u\left(Q(x_j,3r_j)\right)$$
$$\le C\sum_j\left(\frac{1}{(\varphi(|Q(x_j,3r_j)|)|Q(x_j,3r_j)|)^{1-\alpha}}\int_{Q(x_j,r_j)}f(x)dx\right)^{-q}\left(\int_{Q(x_j,r_j)}f(x)^p v(x)\,dx\right)^{\frac{q}{p}}$$
$$\le C\sum_j\frac{1}{(\varphi(|r_x|^n)|r_x|^n)^{1-\alpha}}\left(\int_{Q(x_j,r_j)}f(x)dx\right)^{-q}\left(\int_{Q(x_j,r_j)}f(x)^p v(x)\,dx\right)^{\frac{q}{p}}$$

$$\leq \frac{C}{\lambda^q}\left(\sum_j \int_{Q(x_j,r_j)} f(x)^p v(x)\,dx\right)^{\frac{q}{p}}$$

$$\leq \frac{C}{\lambda^q}\left(\int_{R^n} f(x)^p v(x)\,dx\right)^{\frac{q}{p}},$$

where, we consider that $q/p \geq 1$ and $Q(x_j, r_j)$ are pairwise disjoint.

3. Conclusion

This paper gives the sufficient and necessary conditions for strong and weak type two weight norm inequalities for a new fractional maximal operators by introducing a class of new two weight functions which include classical two weight functions on Lebesgue spaces and our main results will shed some new lights on boundedness of other operators and their commutators on Lebesgue spaces. In addition, we can also consider the two weight characterization of new maximal operators on Morrey spaces.

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