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Post and Jablonsky Algebras of Compositions (Superpositions)

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Abstract: There are two algebras of compositions, Post and Jablonsky algebras. Definitions of these algebras was very simple. The article gives mathematically precise definition of these algebras by using Mal'cev's definitions of the algebras. A. I. Mal'cev defined pre-iterative and iterative algebras of compositions. The significant extension of pre-iterative algebra is given in the article. Iterative algebra is incorrect. E. L. Post used implicitly pre-iterative algebra. S. V. Jablonsky used implicitly iterative algebra. The Jablonsky algebra has the operation of adding fictitious variables. But this operation is not primitive, since the addition of fictitious variables is possible at absence of this operation. If fictitious functions are deleted in the Jablonsky algebra then this algebra becomes correct. A natural classification of closed sets is given and fictitious closed sets are exposed. The number of fictitious closed sets is continual, the number of essential closed sets is countable.

Keywords: Post Algebras, Closed Sets of Functions and Relations, Logic of Superpositions

1. Introduction

One of outstanding achievements of 20th century in mathematics was construction of function closed sets in the algebra of logic by American mathematician E. L. Post ([1], 1921).

This Post achievement was not understood by contemporaries, since it was ahead of time by several decades. Twenty years later, Post gave a more complete description of his achievement [2]. The first response to Post's results appeared in the following year. But it was only in middle of fiftieth that these results get well known.

In the USSR the monograph of Post's results was published in 1966 [3]. The Post results were presented in the monograph in a more accessible form, but from several other positions. This monograph was an impulse for numerous studies in the field of closed sets of functions.

Further development of the Post results was given by the outstanding algebraist A. I. Mal'cev [4]. He constructed two algebras calling pre-iterative and iterative Post algebras The pre-iterative algebra gave a mathematically precise justification of the Post results, iterative algebra gave a mathematically rigorous justification of results of S. V. Jablonsky, G. P. Gavrilov, V. B. Kudryavtsev. Unfortunately, pre-iterative algebra (1976) did not receive due recognition and almost all subsequent researchers ([5], monograph) ignored it.

This paper is devoted to the Post algebra and some of its modifications. It is proved that the Jablonsky algebra is incorrect.

Further, a pre-iterative algebra is called a Post algebra, since E. L. Post implicitly used this algebra. The iterative algebra is called the Jablonsky algebra, since S. V. Jablonsky implicitly used iterative algebra in his writings.

2. Signature and Main Sets of Algebras

The structure of algebras contains the main set and the main operations above this set. Notations of main set and main operations are given by the signature of algebras.

The members of the main set are function names, denoted by $\dot{f}_i^n$, and are relation names, denoted by $\dot{r}_i^n$.

The dot above the f_i^n symbol means that this symbol is a functional constant. The absence of the dot means that this symbol is a functional variable, values of which are names of functions. It is generally accepted to call functional constants and functional variables simply functions and omit the point in the notation. The symbol i is an ordinal number of a function, n is arity of the function. The numeration of functions begins anew for each value of n and similarly for relations.

The signature of algebra variables is x_1, x_2, ... Each symbol f_i^n is associated with word $f_i^n x_1 \ldots x_n$, each word r_i^n is associated with word $r_i^n x_1, \ldots x_n$. Usually brackets and commas are used: $f_i^n(x_1, \ldots, x_n)$, and $r_i^n(x_1, \ldots, x_n)$.

The signature of algebras contains a sort character of values of variables and functions. This sort is denoted by one of characters P (prime numbers), N_k (first k characters of set of natural numbers), N (positive integers), Z (integers), Q (rational numbers), R (real numbers), C (complex numbers).

Each function (relation) is specified by a table. Names of tables are names of functions (relations), the ordinal number of a function (relation) is the ordinal number of their tables.

In the case of sort N_k, the ordinal number of function is a number in k number system, this number is contents of function column when it reads from top to bottom. In the case of the sort N, the ordinal number is in ω number system. The set of these ordinal numbers is countable and can be numbered, but these numbers are very large and can be used in function names only theoretically. In the case of other sorts, the numbers may be ordinals. The names $\dot{f}_i^n$ are abstract, mnemonic names are used instead.

The numbering of relations is also implemented by ordinal numbers by lexico-graphic ordering of rows of tables and by lexico-graphic ordering of tables.

The definition of the signature can be represented by the four $< F, R, X, S >$, where F is the set of symbols $\dot{f}_i^n$, R is the set of symbols $\dot{r}_i^n$, X is the set of symbols x_i, $S \in \{P, N_k, N, Z, Q, R, C\}$.

3. Post Algebra

3.1. Definition of Algebra

A. I. Mal'cev gave the following definition of Post algebra:

Definition Post algebra P_S is

$$P_S = (P_S; \zeta, \tau, \lhd, *)$$

where P_S is the main set of algebra, S is the sort of members of the set, ζ, τ, $\lhd$ and * are the main operations of algebras.

It is generally accepted to denote the sort N_k by k in this definition.

The main operations realize the composition (superposition) of functions and relations. These operations used to construct closed sets of functions and relations. The operations ζ and τ permute variables. The operation $\lhd$ reduces the number of variables in a function and a relation by identifying two variables. The operation * creates new functions and relations by placing a function into a function and relation into relation. The formal definition of these operations is presented below. They are presented with significant additions to the definitions of A. I. Mal'cev. In particular, relations are added to main set, all sorts are used, and constants are added (constats are absent in the Mal'cev definitions).

Main operations are primitives that allow to construct all operations of the algebra P_S. In particular, these operations allow to construct an operation of adding a fictitious variable to a function by placing the projective (selective) function e_2^2 instead of the first variable of any function. The same operations allow to construct an operation of adding a fictitious variable to a relation by placing a two-ary relation with two fictitious variables instead of the first variable of any relation. The proof of these statements is given in section 5.

All Mal'cev's definitions exclude constants. Therefore, constants will be added to the definitions as zero-ary functios. Definitions without constants do not allow the Webb function to generate all functions of the N_k.

3.2. Cyclic Permutation Operation ζ

This operation is an member of the symmetric group. The first variable becomes the last and all variables are shifted left by one position.

Definition One-ary cyclic permutation operation ζ is

$$\zeta f_1^n = f_2^n \leftrightarrow \forall x_1, \ldots, x_n \quad f_2^n(x_1, \ldots, x_n) =$$
$$= f_1^n(x_2, \ldots, x_n, x_1)$$
$$\zeta r_1^n = r_2^n \leftrightarrow n \geq 1 \wedge \forall x_1, \ldots, x_n \quad r_2^n(x_1, \ldots, x_n) =$$
$$= r_1^n(x_2, \ldots, x_n, x_1)$$

In this definition, $\zeta f_1^1 = f_1^1$, $\zeta r_1^1 = r_1^1$ for one-ary functions and relations. For zero-ary functions (constants) $\zeta f_1^0 = f_1^0$, zero-ary relations do not exist. As a result, the operation ζ is performed for all members of the main set.

In the symmetric group, the operation ζ is $\binom{1, \ldots, n-1, n}{2, \ldots, n, 1}$.

The value of the functional variable f_1^n is the name of the function, which is also the name of the table of this function. The ζ operation places the first column of the table after the last column and then shifts all the columns to the left. Similarly for relations.

3.3. The Operation of the Long Permutation τ

This is an another operation of the symmetric group.

Definition Single long permutation operation τ permutes the first and last variables in a function and relation:

$$\tau f_1^n = f_2^n \leftrightarrow \forall x_1, \ldots, x_n \quad f_2^n(x_1, \ldots, x_n) =$$
$$= f_1^n(x_n, x_2, \ldots, x_{n-1}, x_1)$$
$$\tau r_1^n = r_2^n \leftrightarrow n > 0 \wedge \forall x_1, \ldots, x_n \quad r_2^n(x_1, \ldots, x_n) =$$
$$= r_1^n(x_n, x_2, \ldots, x_{n-1}, x_1)$$

In this definition, $\tau f_1^1 = f_1^1$, $\tau r_1^1 = r_1^1$ for one-ary functions and relations. For zero-ary functions $\tau f_1^0 = f_1^0$. Zero-ary relations do not exist. So the operation τ is used for all members of main sets.

In the symmetric group, the operation τ is $\binom{1, 2, \ldots, n-1, n}{n, 2, \ldots, n-1, 1}$.

The operation τ permutes the first and last variable columns in the function and relation tables.

Using the operations ζ and τ, all members of the symmetric group can be get. In this group, the member

$\begin{pmatrix}1,\dots,n\\ i_1,\dots,i_n\end{pmatrix}$ renumbers the variable x_1 to x_{i_1}, the other variables are renumbered in the same way.

3.4. Identification Operation $\lhd$

This operation belongs to the permutation semi-group with repetitions.

Definition The single identification operation $\lhd$ deletes the first variable in the function and in the relation after the identification of the first and second variables:

$$\lhd f_1^n = f_2^{n-1} \leftrightarrow n \geq 2 \wedge \forall x_1, \dots, x_{n-1} f_2^{n-1}(x_1, \dots, x_{n-1}) =$$

$$= f_1^n(x_1, x_1, \dots, x_{n-1})$$

$$\lhd f_1^1 = f_2^1 \leftrightarrow \left(\exists x_1 \exists x_2\ f_1^1(x_1) \neq f_1^1(x_2)\right) \wedge$$

$$\wedge \forall x_1 \quad f_2^1(x_1) = f_1^1(x_1) \quad (essential\ f_1^1)$$

$$\lhd f_1^1 = f_2^0 \leftrightarrow (\forall x_1 \forall x_2 \quad f_1^1(x_1) =$$

$$= f_1^1(x_2)) \wedge f_2^0 = f_1^1(0) \quad (fictitious\ f_1^1)$$

$$\lhd f_1^0 = f_1^0$$

$$\lhd r_1^n = r_2^{n-1} \leftrightarrow n \geq 2 \wedge$$

$$\wedge \forall x_1, \dots, x_{n-1} \quad r_2^{n-1}(x_1, \dots, x_{n-1}) = r_1^n(x_1, x_1, \dots, x_{n-1})$$

$$\lhd r_1^1 = r_1^1$$

This operation deletes rows with different values in the table in the first and second columns for variables. As a result, these columns become equal and the first column is deleted. The arity of function and relation is reduced by one. But this is not possible for zero-ary functions and one-ary relations , so they remain unchanged. This is not possible for a one-ary function too, if it is not fictitious. Indeed, after identification, a single function becomes a constant, and if the function is not fictitious, then it is unclear what value of which row from the function column should become a constant. But with a fictitious function, all the rows of a function column are the same, and the value of any row can become constant.

As a result, the operation $\lhd$ is applicable to all members of main set.

The next subsection will show that only the operation $\lhd$ allows generating zero-ary functions. In particular, the Webb function generates zero-ary functions only by this operation.

In the semi-group with repetitions, the operation $\lhd f^n$ is denoted $\begin{pmatrix}1,2,3,\dots,n\\ 1,1,3,\dots,n\end{pmatrix}$, 1 is repeated here.

*3.5. Substitution Operation * for Functions*

This operation is fundamental in superpositions for functions and relations.

Definition The two-ary substitution operation * replaces the first variable of the function $f_1^{n_1}$ with the function $f_2^{n_2}$:

$$f_1^{n_1} * f_2^{n_2} = f_3^{n_1+n_2-1} \leftrightarrow n_1 \geq 1 \wedge$$

$$\wedge \forall x_1, \dots, x_{n_1+n_2-1} \quad f_3^{n_1+n_2-1}(x_1, \dots, x_{n_1+n_2-1}) =$$

$$= f_1^{n_1}(f_2^{n_2}(x_1, \dots, x_{n_2}), x_{n_2+1}, \dots, x_{n_1+n_2-1}))$$

$$f_1^0 * f_2^{n_2} = f_3^{n_2-1} \leftrightarrow n_2 \geq 1 \wedge$$

$$\wedge \forall x_1, \dots, x_{n_2-1} \quad f_3^{n_2-1}(x_1, \dots, x_{n_2-1}) = f_1^0$$

$$f_1^0 * f_2^0 = f_1^0$$

This operation, together with the operations ζ and τ , performs any substitution.

The operation definition consists of several formulas. In the first formula, there is no substitution in the zero-ary function. In the second formula, the substitution operation is given only to the zero-ary function, but the function $f_2^{n_2}$ should not be zero-ary. The result of the substitution is a function in which all variables are fictitious or absent (if the substitution of a one-ary function is used). The value of this function is the value of the zero-ary function. In both formulas, the resulting function has the four $n_1 + n_2 - 1$. But if $n_1 = n_2 = 0$, then the resulting function has arity 0. This is reflected in the last formula.

As follows from these formulas, constants are generated only by constants. This means that the Webb function cannot generate constants with a substitution operation. Only the identification operation must be used to generate constants.

*3.6. The Substitution Operation * for Relations*

This operation is similar to the operation * for functions, but it has significant limitations.

Definition The two-ary substitution operation * replaces the first variable of a relation $r_1^{n_1}$ with a relation $r_2^{n_2}$:

$$r_1^{n_1} * r_2^{n_2} = r_3^{n_1+n_2-1} \leftrightarrow n_1 \geq 1 \wedge$$

$$\wedge n_2 \geq 1 \wedge \forall x_1, \dots, x_{n_1+n_2-1} r_3^{n_1+n_2-1}(x_1, \dots, x_{n_1+n_2-1})$$

$$= r_1^{n_1}(r_2^{n_2}(x_1, \dots, x_{n_2}), x_{n_2+1}, \dots, x_{n_1+n_2-1}))$$

where

$$r_1^{n_1}\left(r_2^{n_2}(x_1, \dots, x_{n_2}), x_{n_2+1}, \dots, x_{n_1+n_2-1}\right) \leftrightarrow$$

$$\leftrightarrow r_2^{n_2}(x_1, \dots, x_{n_2}) \wedge r_1^{n_1}(x_{n_2}, \dots, x_{n_1+n_2-1})$$

The definition of functions is a special case of this definition.

Indeed, an n -ary function can be represented by a $(n+1)$ -ary relation. In the expression $r_2^{n_2}(x_1, \dots, x_{n_2}) \wedge r_1^{n_1}(x_{n_2}, \dots, x_{n_1+n_2-1})$ with the value of the function $f_2^{n_2-1}$ is x_{n_2}, the value of the function $f_1^{n_1-1}$ is $x_{n_1+n_2-1}$. After substitution, the first variable in $r_1^{n_1}$ (and the first variable in $f_1^{n_1-1}$) is the value of the function $f_2^{n_2-1}$, that is, x_{n_2}.

The operation *, in conjunction with the operations ζ and τ, performs any substitution.

3.7. Closing Operation

A. I. Mal'cev gave the definition of this operation in addition to the definition of Post algebra.

The definitions of main operations of Post algebra establish the rules for constructing new functions. These rules and the definition of the signature of algebras belong to logic; they apply to many theories. The definition of a closure operation does not belong to logic.

Definition Let F_1 be some set of functions (relations). The closure of F_1 is the set $[F_1]$ containing

- functions (relations) from F_1,

- the result of applying the operations of cyclic permutation, long permutation and identification to functions (relations) from F_1,

- is the result of substituting a function (relation) from $[F_1]$ into a function (relation) from $[F_1]$:

$$[F_1] = F_2 \leftrightarrow (\forall f_1 \in F_1 \ \ f_1 \in F_2) \wedge$$

$$\wedge (\forall f_1 \in F_2 \ \ \zeta f_1 \in F_2 \wedge \tau f_1 \in F_2 \wedge \lhd f_1 \in F_2) \wedge$$

$$\wedge \forall f_1, f_2 \in F_2 \ \ f_1 * f_2 \in F_2$$

$$[F_1] = F_2 \leftrightarrow (\forall r_1 \in F_1 \quad r_1 \in F_2) \wedge$$

$$\wedge (\forall r_1 \in F_2 \quad \zeta r_1 \in F_2 \wedge \tau r_1 \in F_2 \wedge \lhd r_1 \in F_2) \wedge$$

$$\wedge \forall r_1, r_2 \in F_2 \ \ r_1 * r_2 \in F_2$$

This definition is iterative. The first step of the iteration F_2 gives all functions (relations) of F_1. The second step adds the superpositions of functions (relations) from F_2 to F_2. The next step adds the compositions of functions (relations) of F_2 to F_2. And so on.

4. Jablonsky Algebra

A. I. Mal'cev gave the following definition of this algebra.

Definition Jablonsky algebra P S is

$$P_S = (P_S; \zeta, \tau, \lhd, \rhd, *)$$

where P_S is the main algebra set, ζ, τ, $\rhd$, $\lhd$ and * are the main operations of the algebra.

This algebra, unlike the previous one, has another main operation of adding a fictitious variable to a function and relation.

Definition A variable x_i is fictitious if

$$\forall x_i, x_{i'} \quad x_i \neq x_i' \rightarrow f^n(x_1, \dots, x_i, \dots, x_n) =$$

$$= f^n(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$$

$$\forall x_i, x_{i'} \quad x_i \neq x_i' \rightarrow r^n(x_1, \dots, x_i, \dots, x_n) =$$

$$= r^n(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$$

Functions and relations are called *fictitious* or *essential*, if they contain or do not contain fictitious variables.

Definition The operation $\rhd$ of adding fictitious variable is

$$\rhd f_1^n = f_2^{n+1} \leftrightarrow \forall x_1, \dots, x_{n+1} f_2^{n+1}(x_1, \dots, x_{n+1})$$

$$= f_1^n(x_2, \dots, x_{n+1})$$

$$\rhd r_1^n = r_2^{n+1} \leftrightarrow \forall x_1, \dots, x_{n+1} r_2^{n+1}(x_1, \dots, x_{n+1})$$

$$= r_1^n(x_2, \dots, x_{n+1})$$

A. I. Mal'cev designated this operation as ∇. This designation is overloaded. New designations are more mnemonic: $\rhd$ indicates an increase in the number of variables, $\lhd$ indicates a decrease in the number of variables (A. I. Mal'cev used Δ instead of $\rhd$ to identify the variables).

According to the operation $\rhd$, each essential function (relation) in a closed set has added the infinite set of functions (relations) with fictitious arguments. This means that finite closed sets of any functions (relations) do not exist. But finite closed sets exist in Post algebra and there exist infinite closed sets, which contain only essential functions (relations).

The Jablonsky algebra is a subalgebra of Post algebra, since all closed sets of the Jablonsky algebra are present in Post algebra.

5. Jablonsky Algebra Is Correct

An operation is primitive if it is not constructed by other operations. Operations of composition algebras must be primitive. Jablonsky algebra is incorrect since its operation of adding fictitious variables is not primitive.

Theorem The $\rhd$ operation is not primitive.

Proof. Only substitution by a two-ary function can increase arity of any function and relation by one, viz, can add a variable.

A variable added to a function f will be first and fictitious if the first variable of f is substituted by a two-ary function, which have the first variable fictitious. This two-ary function is only the projective (selective) function $e_2^2(x_1, x_2)$. The substitution of $e_2^2(x_1, x_2)$ in f adds a fictitious variable to f, since $f(e_2^2(x_1, x_2), x_3, \dots, x_{n+1}) = f(x_2, x_3, \dots, x_{n+1})$, viz, the result of the substitution does not depend on value of the first variable.

The variable added to the relation r_1^n will be first and fictitious if (1) the substituted two-ary relation r_2^2 has both variables fictitious, (2) the sort of the first of these variables and the sort of variable that is supposed to be added in r_1^n are the same, (3) the sort of the second variable in r_2^2 and the sort of the first variable in r_1^n are the same, (4) the relation r_2^2 is placed instead of the first variable of r_1^n.

The substitution r_2^2 in r_1^n, viz, $r_1^n(r_2^2(x_1, x_2), x_3, \dots, x_{n+1})$, really adds the fictitious first variable in r_1^n, if the first variable is fictitious in r_2^n and the second variable contains only values coinciding with values of the first variable in r_1^n.

Therefore, the operation of adding a fictitiousy variable is not primitive.

The Jablonsky algebra is incorrect, but its results are **correct**, since this algebra is isomorphic to a new algebra whose basic set does not contain fictitious functions.

The new algebra is correct, since there is no operation adding a fictitious variable. If fictitious functions in closed sets of the Jablonsky algebra are removed, then all closed sets in the new algebra are get. This is the great value of Jablonsky algebra.

Fictitious objects of any theory are needed only to construct a classification of all objects. In the future, all fictitious objects are removed from the theory. The new algebra does not contain fictitious functions and relations, but not all fictitious objects are removed from it.

6. Fictitious Closed Sets

Only Post algebra will be considered. The obtained results can be applicable to other algebras. They can be applicable to closed sets of relations, too, since the operations of function compositions are a special case of operations of relation compositions. The sort N_k, can be considered but the main obtained results are valid for other sorts too.

The main obtained results are next: the number of fictitious (useless) closed sets is continuous, and the number of essential closed sets is countable. This means that the algebras of compositions of many valued functions and relations do not contain anything essentially new to compare with algebras of two-valued functions and relations [6,7].

6.1. Bases of Closed Sets

Almost every infinite closed set has a continual number of sets generating it. Therefore, it is common to use generating sets that are bases.

Definition A set of functions that generates a closed set is a basis if any proper subset of these functions does not generate this closed set.

But almost any infinite closed set of functions has countable set of bases. Therefore, the minimal basis must be chosen.

Definition A basis is minimal if the number of functions in it is minimal. If the number of functions in bases is the same, then a basis with minimal lexico-graphical order must be chosen. In this case, the functions in each basis must be arranged in a sequence of their decreasing numbers. The basis is minimal if it has functions with minimal numbers.

There are closed sets that have no bases [8], and there are closed sets that have only infinite bases (bases with an infinite number of functions). As a rule, such sets have a single infinite basis.

6.2. Classification of Closed Sets

The main problem of each theory is the classification of its objects by using properties obtained in theorems of the theory. And each object should belong to only one class. Such a classification is called *natural*.

There are many papers devoted to the classification of closed sets, but only one of them [9] gives a natural classification of these sets. This classification uses the number of functions in a minimal basis.

Definition The class U_0 contains closed sets without a basis. The class U_m contains closed sets, the minimal basis of which has m functions. The class U_ω contains closed sets with an infinite basis.

The class U_0 is possibly finite. Each class U_m with $1 \leq m < \omega$ is countable, since the set of all functions of type N_k is countable. The class U_ω is continual [8].

6.3. Fictitious Closed Sets

As mentioned above, the main problem of any theory is the natural classification of objects of this theory. The next main problem is the identification and deliting of fictitious objects. As a rule, the number of fictitious objects is incomparably greater than the number of essential objects.

Theorem A closed set is fictitious (useless) for classification if it belongs to the class U_m with $m \neq 1$.

Proof. Each function generates the closed set. A set of functions, that generates a closed set, is called class. Then any function belongs to a unique class, and each class belongs to a unique closed set of U_1. Consequently, a natural classification of all functions has obtained, viz, classification of all objects of the main set. This means that closed sets U_m with $m \neq 1$ are useless to classify functions.

Thus, the number of essential closed sets is countable, and the number of fictitious closed sets is continual. As a result, the second main problem of theories is completed and huge number of fictitious objects is deleted.

An algebra with fictitious closed sets is fictive. Fictive algebras have the following property.

Theorem A fictive algebra becomes empty after removing all essential algebras from it.

Proof. By removing one essential algebra, one function in the minimal basis of this algebra is removed. By removing all essential algebras, the basis becomes empty. But empty basis has empty set.

Any essential algebra is not empty after deleting the other essential algebras since it bases are not deleted.

A fictitious algebra is the union of essential algebras. Any essential algebra is not a union of essential algebras. An essential algebra is not union of its algebras, since the union of these algebras is the new algebra. The new algebra contains the essential algebra.

It is necessary to emphasize the distinction between fictitious functions and fictitious algebras. There are fictitious algebras of essential functions, and there are essential algebras of fictitious functions.

The classification of algebras given above has only one level. Usually, the classification contains many levels, in particular, the Post classification [2] contains an infinite number of levels. The main result is that the multi-level classification should be built only for closed classes of U_1.

As an example, a multilevel classification of fictitious closed classes of Boolean functions was constructed in [10]. This example demonstrates useless of such classification.

7. Conclusions

Mathimically precise construction of algebras of compositions is given. The signature of the algebras contains all sorts of objects of the algebras. The sorts are changed from prime numbers to complex numbers. The objects of algebras are functions and relations. Functions and relations can be any valued. Mal'cev's definitions of all algebras of composition are extended by the signature and are added by zero-ary functions (constants). So the algebras become complete. A. I.

Mal'cev defined two algebras of compositions, one of them was used implicitly by E. L. Post and the other was used implicitly by S. V. Jablonsky. It is shown that Jablonsky algebra is not correct, and the way to correct the algebra is pointed.

The classification of close sets of functions in both algebras uses bases. Any class of the classification has close sets generated by bases contained the same number of functions. This number is changed from zero to infinity. The class of close sets generated by infinite basis has infinite set of subclasses. Any other class has countable set of subclasses. But all classes are fictitiouse (useless) for classification of functions. Exseption is the class of closed sets generated by unit bases. This class is essential. Then the number of fictitious closed sets is countinuous, the number of essential closed sets is contable.

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