Pure and Applied Mathematics Journal
2017; 6(3): 101-107
http://www.sciencepublishinggroup.com/j/pamj
doi: 10.11648/j.pamj.20170603.12
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Remarks on *A*-skew-adjoint, *A*-almost Similarity Equivalence and Other Operators in Hilbert Space

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To cite this article:
Isaiah Nalianya Sitati, Bernard Nzimbi, Stephen Luketero, Jairus Khalagai. Remarks on *A*-skew-adjoint, *A*-almost Similarity Equivalence and Other Operators in Hilbert Space. *Pure and Applied Mathematics Journal*. Vol. 6, No. 3, 2017, pp. 101-107.
doi: 10.11648/j.pamj.20170603.12

Received: April 28, 2017; **Accepted**: May 9, 2017; **Published**: June 29, 2017

Abstract: In this paper, notions of A-almost similarity and the Lie algebra of A-skew-adjoint operators in Hilbert space are introduced. In this context, A is a self-adjoint and an invertible operator. It is shown that A-almost similarity is an equivalence relation. Conditions under which A-almost similarity implies similarity are outlined and in which case their spectra is located. Conditions under which an A-skew adjoint operator reduces to a skew adjoint operator are also given. By relaxing some conditions on normal and unitary operators, new results on A -normal, binormal and A-binormal operators are proved. Finally A-skew adjoint operators are characterized and the relationship between A-self- adjoint and A-skew adjoint operators is given.

Keywords: Skew-adjoint, A-skew-adjoint, A-almost Similarity, Hilbert Space, A-Normal and Binormal

1. Introduction

In this paper, Hilbert space(s) or subspace(s) will be denoted by capital letters, H and K respectively and T, A, B etc denote bounded linear operators. In this context, an operator will mean a bounded linear transformation. $B(H)$ will denote the Banach algebra of bounded linear operators on a Hilbert space H and $B(H, K)$ denotes the set of bounded linear transformations from one Hilbert space H to another one K, which is equipped with the (induced uniform) norm. Hilbert space operators have been discussed by many others like [5], [16], [18] and [20] among other scholars.

If $T \in B(H)$, then T^* denotes the adjoint while $Ker(T)$, $Ran(T)$, $\overline{M}$ and $M^{\perp}$ stands for the kernel of T, range of T, closure of M and orthogonal complement of a closed subspace M of H respectively. For an operator T, we also denote by $\sigma(T)$, $\| T \|$ the spectrum and norm of T respectively. A contraction on H is an operator $T \in B(H)$ such that $T^*T \leq I$ (i.e. $\| Tx \| \leq \| x \|\ \forall x \in H$). A *strict or proper* contraction is an operator T with $T^*T < I$ (i.e. $\underset{0 \neq x}{Sup} \frac{\|Tx\|}{\|x\|} < 1$). If $T^*T = I$, then T is *called a non-strict contraction* (or an *isometry*). Many authors like Kubrusly [5] and Nzimbi et al [10] have extensively studied this class of operators.

An operator $T \in B(H)$ is said to be positive if $\langle Tx, x \rangle \geq 0\ \forall x \in H$. Suppose that $A \in B(H)$ is a positive operator, then an operator $T \in B(H)$ is called an

$A - contraction$ on H if $T^*AT \leq A$. If equality holds, that is $T^*AT = A$, then T is called an $A - isometry$, where A is a self adjoint and invertible operator.(See more of contractions in [4] and [17]).

In this research, we put more conditions on A. In particular, if A is a self adjoint and invertible operator, then we call such an $A - isometry$ an $A - Unitary$. Let T be a linear operator on a Hilbert space H. We define the $A - adjoint$ of T to be an operator S such that $AS = T^*A$ whose existence is not guaranteed. It may or may not exist. In fact a given $T \in B(H)$ may admit many A-adjoints and if such an $A - adjoint$ of T exists, we denote it as $T^{[*]}$. Thus $AT^{[*]} = T^*A$. As it were before, A is invertible and so $T^{[*]} = A^{-1}T^*A$. It is also clear that $A - adjoint$ of T is the adjoint of T if $T = I$. Earlier results proved by Kubrusly [5] have shown that, T admits an $A - adjoint$ if and only if $Ran(T^*A) \subset Ran(A)$. In this case the operator A is acting as a signature operator on H.

Two operators $T \in B(H)$ *and* $S \in B(K)$ *are similar (denoted* $T \approx S$) if there exists an operator $X \in$

$\mathcal{G}\,(H,K)$ where $\mathcal{G}\,(H,K)$ is a Banach subalgebra of $B\,(H,K)$ which is an invertible operator from H to K such that

$$XT = SX \text{ (i. e, } X^{-1}SX \text{ or } S = XTX^{-1}).$$

$T \in B(H)$ *and* $S \in B(K)$ are unitarily equivalent (denoted $T \cong S$), if there exists a unitary operator $U \in \mathcal{G}\,(H,K)$ such that $UT = SU$

$$(i.e, T = U^*SU \text{ or equivalently } S = UTU^*).$$

Two operators are considered the "same" if they are unitarily equivalent since they have the same, properties of invertibility, normality, spectral picture (norm, spectrum and spectral radius).

Two linear operators $T \in B(H)$ and $S \in B(K)$ are said to be $A - unitarily$ equivalent (denoted $T \cong S$), if there exists an $A - unitary$ operator $U \in \mathcal{G}\,(H,K)$ such that $TU = US$.(For more details on this equivalence see [9] and [12]).

The following classes of bounded linear operators shall be defined in this paper:

An operator $T \in B(H)$ is said to be:

self- adjoint or Hermitian if $T^* = T$ (equivalently, if $\langle Tx, x\rangle\ \forall\ x \in H$)

A projection if $T^2 = T$ and $T^* = T$

unitary if $T^*T = TT^* = I$

isometric if $TT^* = I$

self-adjoint unitary or a symmetry if $T = T^* = T^{-1}$.

normal if $T^*T = TT^*$ (equivalently, if $\| Tx \| = \| T^*x \|$ $\forall\ x \in H$),

binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$

Let H and K be Hilbert spaces. An operator $X \in B(H,K)$ is invertible if it is injective (one -to- one) and surjective (onto or has dense range); equivalently if $Ker(X) = \{0\}$ and $\overline{Ran(X)} = K$. We denote the class of invertible linear operators by $\mathcal{G}\,(H,K)$.

The commutator of two operators A and B, denoted by $[A,B]$ is defined by $AB - BA$. The self –commutator of an operator A is $[A, A^*] = A^*A - AA^*$.

Suppose $A \in B(H)$ is a self- adjoint and invertible operator, not necessarily unique. An operator $T \in B(H)$ is said to be:

$A - self\ adjoint$ $if\ T^* = ATA^{-1}$ (equivalently, $T^{[*]} = T$).

$A - skew - adjoint\ if\ T^* = -ATA^{-1}$ (equivalently, $T^{[*]} = -T$).

$A - normal$ if $A^{-1}T^*AT = TA^{-1}T^*A$ or equivalently, $T^{[*]}T = TT^{[*]}$.

$A - unitary$ if $T^*AT = A$ or equivalently, $T^{[*]} = T^{-1}$.

$A - binormal$ if $[T^{[*]}T, TT^{[*]}] = 0$.

It has to be noted that an A-isometry whose range is dense in H is an $A - unitary$.

2. Basic Results

Definition 2.1: Let H denote a Hilbert Space and $B(H)$ denote the Banach algebra of bounded linear operators. Two operators $A \in B(H)$ *and* $B \in B(K)$ *are similar* (denoted by $A \approx B$) if there exists an invertible operator $N \in \mathcal{G}\,(H,K)$ where $\mathcal{G}\,(H,K)$ is a Banach subalgebra of B (H,K) which is an invertible operator from H to K such that $NA = BN$ or equivalently $A = N^{-1}BN$ or $B = NAN^{-1}$.

Two operators A and B in $B(H)$ are said to be almost similar (a.s) (denoted by $A \overset{a.s}{\sim} B$) if there exists an invertible operator N such that the following two conditions are satisfied:

$$A^*A = N^{-1}(B^*B)N$$

$$A^* + A = N^{-1}(B^* + B)N.$$

It has already been shown by many authors like [9], [12] and [13] that similarity, almost similarity and unitary equivalences are equivalence relations.

Remark 2.2: It has to be noted that almost similarity generally does not imply similarity. However, certain conditions can guarantee this preservation. These may include the following:

Theorem 2.3 [13]: If $T \in B(H)$ and $S \in B(H)$ are almost similar projection operators, then $\sigma(T) = \sigma(S)$.

Proof: See [13].

Corollary 2.4 [13]: If $T \in B(H)$ and $S \in B(H)$ are almost similar self-adjoint operators, then they are similar.

Proof: See [13].

Corollary 2.5 [13]: If $T \in B(H)$ and $S \in B(H)$ are almost similar self-adjoint operators, then $\sigma(T) = \sigma(S)$.

Remark 2.6: Equality of spectrum does not in general imply similarity of operators T and S unless if their multiplicities are the same. As an example if we consider operators T and S as $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Clearly, both these operators are self-adjoint. However, by computation we see that they are not almost similar because $\sigma(T) \neq \sigma(S)$, that is $\sigma(T) = \{-1,1,1\} \neq \{-1,-1,1\} = \sigma(S)$ and so their multiplicities do not coincide.

Theorem 2.6 [10]: $T \in B(H)$ is A-self adjoint, if and only if is $AT -$ self adjoint.

Proof: See [10].

Corollary 2.8 [10]: Let $T \in B(H)$ be A-self- adjoint, T is self- adjoint if $AT = TA$.

Proof: See [10].

3. *A*-Almost Similarity of Operators

Definition 3.1: Let $T \in B(H)$ and $S \in B(H)$. Then T is said to be $A - almost\ similar$ to S (denoted by $T \overset{a.a.s}{\sim} S$) if there exists an invertible operator $N \in B(H,K)$ such that the following two conditions are satisfied:

$$T^{[*]}\,T = N^{-1}\big(S^{[*]}S\big)N$$

$$T^{[*]} + T = N^{-1}\big(S^{[*]} + S\big)N,$$

where $T^{[*]} = T$ and $S^{[*]} = S$ are A-self-adjoint operators, that is $T = A^{-1}T^*A$ and $S = A^{-1}S^*A$ respectively.

Theorem 3.2: $A - almost\ similarity$ is an equivalence relation.

Proof: It is shown that this relation is reflexive, symmetric and transitive.

Reflexivity: Let $T \in B(H)$. Then $T^{[*]}\,T = N^{-1}\big(T^{[*]}\mathrm{T}\big)N$ where N is an invertible operator.

It is also clear that $T^{[*]} + T = N^{-1}\big(T^{[*]} + \mathrm{T}\big)N$. Hence $T \overset{a.a.s}{\sim} T$. In this case we can choose (w.l.o.g) $N = 1$.

Next symmetry is established, that is if $T \overset{a.a.s}{\sim} S \implies S \overset{a.a.s}{\sim} T$. Suppose that $T \overset{a.a.s}{\sim} S$, there exists an invertible operator N such that

$$T^{[*]}\,T = N^{-1}\big(S^{[*]}\mathrm{S}\big)N \tag{1}$$

and

$$T^{[*]} + \,T = N^{-1}\big(S^{[*]} + \mathrm{S}\big)N \tag{2}$$

Pre-multiplication of (1) and (2) by N and post multiplication of (1) and (2) by N^{-1} and applying the adjoint operation gives

$S^{[*]}\,S = M^{-1}\big(T^{[*]}T\big)M$ and

$S^{[*]} + S = M^{-1}\big(T^{[*]} + T\big)M$ where $N\ = M^{-1}$ which is an invertible operator, since N^{-1} is invertible. Hence $S \overset{a.a.s}{\sim} T$.

Finally, let T, S and Q be in $B(H)$. Suppose that $T \overset{a.a.s}{\sim} S$ and $S \overset{a.a.s}{\sim} Q$. It then follows that

$$T^{[*]}\,T = N^{-1}\big(S^{[*]}\mathrm{S}\big)N, T^{[*]} + \,T = N^{-1}\big(S^{[*]} + \mathrm{S}\big)N \tag{3}$$

and

$$S^{[*]}\,S = M\big(Q^{[*]}\mathrm{Q}\big)M, S^{[*]} + \,S = M^{-1}\big(Q^{[*]} + \mathrm{Q}\big)M, \tag{4}$$

where M and N are invertible operators. Using (3) and (4) it is found out that

$T^{[*]}T = N^{-1}\big[M^{-1}\big(Q^{[*]}Q\big)M\big]N = (MN)^{-1}Q^{[*]}Q(MN) = X^{-1}(Q^{[*]}Q)X$ and

$T^{[*]} + T = N^{-1}\big[M^{-1}\big(Q^{[*]} + Q\big)M\big]N = (MN)^{-1}Q^{[*]} + Q(MN) = X^{-1}(Q^{[*]} + Q)X$ where $X = MN$, is invertible (since M and N are invertible). Hence $T \overset{a.a.s}{\sim} Q$ which proves transitivity.

Corollary 3.3: If $T \in B(H)$ and $S \in B(H)$ are projection operators such that $T\ is\ A - almost\ similar$ to S then T is similar to S. Moreover, $\sigma_p(T) = \sigma_p(S)$.

Proof: By definition of $A - almost\, similarity$ there exists an A-invertible operator N such that

$$T^{[*]}\,T = N^{-1}\big(S^{[*]}\mathrm{S}\big)N \tag{5}$$

and

$$T^{[*]} + \,T = N^{-1}\big(S^{[*]} + \mathrm{S}\big)N \tag{6}$$

From (5) using the definition of A-Self adjoint of an operator we have, $A^{-1}T^*AT = N^{-1}[A^{-1}S^*AS]N$, that is $A^{-1}T^*TA = N^{-1}[A^{-1}S^*SA]N$ (by [10, corollary 3.14] where $AT = TA$ and $AS = SA$ respectively). It follows that $A^{-1}T^2A = N^{-1}[A^{-1}S^2A]N$ (since T and S are projection operators) i.e. $A^{-1}TA = N^{-1}[A^{-1}SA]N$, i.e., $A^{-1}AT = N^{-1}[A^{-1}AS]N$ (by [10, corollary 3.14]) i.e. $T = N^{-1}SN$ and so $T{\sim}S$.

In like manner, from (6) $A^{-1}T^*A + T = N^{-1}[A^{-1}S^*A + S]N = N^{-1}[\,A^{-1}S^*A]N + N^{-1}SN$, that is

$A^{-1}TA + T = N^{-1}[\,A^{-1}SA]N + N^{-1}SN$ i.e $A^{-1}AT + T = N^{-1}[\,A^{-1}AS]N + N^{-1}SN$ (by [10, corollary 3.14]), that is $T + T = N^{-1}(S + S)N$ i.e $2T = N^{-1}(2S)N \implies T = N^{-1}SN$ and so $T{\sim}S$. But similar operators have the same point spectrum. Hence, $\sigma_p(T) = \sigma_p(S)$ as required.

Remark 3.4:

a) The above corollary gives a condition under which $A - almost\ similarity \implies Similarity$ of operators.
b) Conditions imposed on operators S and T so that they have the same spectrum is that they should both be projections, that is $S = S^*$; $T = T^*$ and $S^2 = S$; $T^2 = T$.

4. *A*-Skew Adjoint and *A*-Normal Operators

In this section, some properties of the Lie algebra $\mathbb{L}_A\ of\ A - skew - adjoint$ operators are outlined. The following basic definitions are of essence:

Definition 4.1: A Lie algebra is a vector space $\mathbb{L}$ over some field $\mathbb{F}$ together with a binary operation $[,]: \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ called the Lie bracket such that

1) $[,]$ is bilinear that is $[\,ax + by, z\,] = a[\,x, y\,] + b[\,y, z\,]$ and $[z, ax + by\,] = a[\,z, x] + b[z, y\,]$ $\forall\ a, b\ \in \mathbb{F}$ and $x, y, z\ \in \mathbb{L}$
2) $[\,x, x\,] = 0$ or $[\,x, y\,] = -[y, x\,]\ \forall\ x, y \in \mathbb{L}$
3) $\big[\,x, [\,y, z\,]\big] + \big[\,y, [\,z, x\,]\big] + \big[\,z, [\,x, y\,]\big]$=0 $\forall\ x, y, z \in \mathbb{L}$. This is called the Jacobi identity.

Example 4.2: Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$. Let $\mathbb{L} = End_{\mathbb{F}}\mathbb{V}$ i.e the endomorphism of the vector space $\mathbb{V}$ over the field $\mathbb{F}$, that is linear maps from $\mathbb{V}$ to $\mathbb{V}$. Alternatively we may take (for finite dimensional $\mathbb{V}$) the set of all $n \times n$ matrices (operators). As usual, define on $[\,A, B\,] = AB - BA$, $\forall\ A, B\ \in End_{\mathbb{F}}\mathbb{V}$. Then $[A, A] = 0$, $[\,A, B\,] = AB - BA =$

$-(BA - AB) = -[B, A]$ and $[\,A, [B, C]\,] + [\,B, [C, A]\,] + [\,C, [A, B]\,] = 0$.

Note that if $\mathbb{V}$ is n-dimensional, then $End_{\mathbb{F}}\mathbb{V}$ is n^2 −dimensional vector space over $\mathbb{F}$. This Lie algebra is called a Linear Lie algebra over $\mathbb{F}$.

Another example of a Lie algebra could be the ordinary vectors in three dimensions. They form a three dimensional vector space over a field $\mathbb{R}$. Define $[\,a, b] = a \times b$ to be the usual vector cross product. Then, by computation it is seen that

$[\,a, a] =\ a \times a = 0$, $[a, b] = -[b, a] = a \times b$ and
$a \times (b \times c) +\ b \times (c \times a) + c \times (a \times c) = 0\ a, b, c\ \in \mathbb{L}$.

Remark 4.3: Denote by

1) The Lie algebra $\mathbb{L}_A\ of\ A - skew - adjoint$ operators is the set

$$\mathbb{L}_A = \{T \in B(H): T^{[*]}\ = -T\}$$

2) The Jordan algebra $\mathbb{J}_A$ of $A - self\ adjoint$ operators is the set

$\mathbb{J}_A = \{T \in B(H): T^{[*]} = T\}$. Note that just like the Lie algebra $\mathbb{L}_A$, the Jordan algebra $\mathbb{J}_A$ is an $\mathbb{R}$- linear subspace. That is, it is closed under real linear combinations. (See more results of this class of operators in [2] and [10].

Remark 4.4: For A-self-adjoint and A-skew-adjoint operators, multiplication is not preserved at all. It has also to be noted that every $T \in B(H)$ admits a Cartesian decomposition $T = ReT + iImT$, where $Re\,T = \frac{1}{2}(T + T^*)$ and $Im\,T = \frac{1}{2}(T - T^*)$. In addition to this, if T is A-skew-adjoint, then its adjoint T^* is also A-skew-adjoint.

The following results form a basis of discussion in this section and are stated without proof:

Theorem 4.5 [10]: $T \in B(H)$ is skew-adjoint if $ReT = 0$.

Theorem 4.6 [10]: Every skew-adjont operator T is A-skew-adjoint.

Corollary 4.7 [10]: If $T \in B(H)$ is A-skew-adjoint, then T is skew-adjont if and only if $[T, A] = 0$.

Remark 4.8: The above corollary is a condition which guarantees an A-skew-adjoint to be skew-adjoint. But $T \in B(H)$ admits a Cartesian decomposition as illustrated in Remark 4.4. It is clear that this operator $T \in B(H)$ is skew-adjoint if $Re\,T = 0$. Now, some general results for an A-skew-adjoint operator $T \in B(H)$ which follow immediately from this remark are given:

Theorem 4.9: $T \in B(H)$ is A −skew-adjoint if $ReT = 0$.

Proof: Every $T \in B(H)$ is a linear combination of self adjoint operators, that is

$T = T_1 + iT_2$. But $T \in B(H)$ is A −skew-adjoint and so $T^{[*]} = -T$, that is

$T^* = -ATA^{-1} = -A(T_1 + iT_1)A^{-1} = -AT_1A^{-1} - iAT_2A^{-1}$. By the above remark $T \in B(H)$ is A −skew-adjoint implies that $T \in B(H)$ is skew-adjoint if and only if $AT = TA$ that is $[T, A] = 0$.

Thus $T^* = -T_1AA^{-1} - iT_2AA^{-1} = -T_1 - iT_1$.

But $T^* = (T_1 + iT_1)^* = T_1 - iT_1$ (since T_1 and T_2 are commuting self-adjoint operators). That is $T_1 - iT_1 = -T_1 - iT_1$. Validity of this equality is guaranteed if and only if $T_1 = 0$ which is $ReT = 0$.

Theorem 4.10: Let A be a symmetry. If T is A-skew-adjoint, then $T^*$is A-skew-adjoint.

Proof: By definition, T is A-skew-adjoint means $T = -A^{-1}T^*A$, that is $T^{[*]} = -T$ and so $T^* = -ATA^{-1}$. Taking adjoints on both sides of this equation gives

$$(T^*)^* = (-ATA^{-1})^* = A^*T^*(A^{-1})^*$$

i.e. $T = -A^{-1}T^*(A^{-1})^{-1}$ (Since A is a symmetry)

i.e. $T = -A^{-1}T^*A$ (or equivalently $T = -T^{[*]}$). Therefore $T^*$is A-skew-adjoint as required.

Proposition 4.11 [10]: Every skew-adjont operator $T \in B(H)$ is normal.

Proof: (See [10]).

Proposition 4.12: Let $T \in B(H)$ be an A-skew-adjoint such that $[T, A] = 0$. Then T is normal.

Proof: By definition, $T^* = -ATA^{-1}$. Simply check whether T and T^* commute, that is

$T^*T = -ATA^{-1}T = -TAA^{-1}T$ (Since $[T, A] = 0$) i.e. $T^*T = -T^2$.

Similarly $TT^* = -TATA^{-1} = -TTAA^{-1} = -T^2$. From the right hand side of these two equations, It is then established that $[T, T^*] = 0$. Therefore T is normal.

Corollary 4.13: Let $T \in B(H)$ be an A-skew-adjoint such that $[T, A] = 0$. Then T is A-normal.

Proof: Given $T \in B(H)$ an A-skew-adjoint, then $T^* = -ATA^{-1}$. It is sufficient enough to show that $[T^{[*]}, T] = 0$. But A-normal means $A^{-1}T^*AT = TA^{-1}T^*A$. Since T is A-skew-adjoint, replacing $T^* = -ATA^{-1}$ in this equation yields

$-A^{-1}ATA^{-1}AT = -TA^{-1}ATA^{-1}A$, i.e $-T^2 = -T^2 \Leftrightarrow T^2 = T^2$.

This means that $[T^{[*]}, T] = 0$, so T is A-normal as required.

Theorem 4.14: Suppose T and S are commuting A-skew-adjoint operators. Then T^* and S^* commute.

Proof: By definition of A-skew-adjointness, $T^* = -ATA^{-1}$ and $S^* = -ASA^{-1}$. In addition, the operators S and T are also commuting, that is $ST - TS = 0$. It has to established that $[T^*, S^*] = 0$.

Thus, $(ST - TS)^* = T^*S^* - S^*T^*$

$=(-ATA^{-1})(-ASA^{-1}) - (-ASA^{-1})(-ATA^{-1})$

$=ATA^{-1}ASA^{-1} - ASA^{-1}ATA^{-1}$

$=ATSA^{-1} - ASTA^{-1} = A(TS - AST)A^{-1} = 0.$

From this, it is concluded that T^* and S^* commute, that is $[T^*, S^*] = 0$.

Remark 4.15: For an operator $T \in B(H)$ which is both A-skew-adjoint and A −unitary, its spectrum can be found from its Eigen values. A quick computation shows that

$T + T^{-1} = 0$. Thus for any $0 \neq x \in H$, it follows that

$$0=\langle (T + T^{-1})x, x\rangle = \langle Tx, x\rangle + \langle T^{-1}x, x\rangle = \langle \lambda x, x\rangle + \left\langle \frac{1}{\lambda}x, x\right\rangle = \lambda\langle x, x\rangle + \frac{1}{\lambda}\langle x, x\rangle$$

$$= (\lambda^2 + 1)\langle x, x\rangle = 0, \text{ that is } \lambda^2 + 1 = 0 \text{ i.e, } \lambda = \pm i.$$

From this result, it is seen that $\sigma(T) \subseteq \{i, -i\}$.

Note also that if T is A-self-adjoint, then T and T^* are similar and hence have the same spectrum. However, this is not always the case for an A-normal operator. This is illustrated in the example below:

Example 4.16: Consider T to be a diagonal operator $\{\omega_i\}$. Denote the adjoint of T by

$T^* = \overline{\{\omega_\iota\}}$. Without loss of generality, let $T = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$. Clealy, $\sigma(T) = \{i, -i\}$ and

$\sigma(T^*) = \{-i, -i\}$. So T and T^* are not similar although T being normal implies that T is A − normal. Letting $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then it is evident that T is A-normal, A-unitary, and A-skew-adjoint. Additionally, T is unitary and skew adjoint. However, it is not A-self-adjoint (and hence not self-adjoint). Hence there are some operators which are skew-adjoint and not A-self-adjoint.

Theorem 4.17 [10]: Let $T = T_1 + iT_2$ be a decomposition T, where T_1 and T_2 are commuting A-self-adjoint operators. Then T is A-normal.

Proof: (See [10]).

Theorem 4.18: Let $T = T_1 + iT_2$ be a decomposition T, where T_1 and T_2 are commuting A-self-adjoint operators. Then T is A −binormal.

Proof: Given $T = T_1 + iT_2$, by the definition of the adjoint of an operator T, then

$T^* = T^*{}_1 - iT^*{}_2$. It suffices to show that $[T^{[*]}T, TT^{[*]}] = 0$ i.e.

$$(T^{[*]}T)(TT^{[*]}) = (TT^{[*]})(T^{[*]}T).$$

$$\text{Now } (T^{[*]}T)\big(TT^{[*]}\big) = (A^{-1}T^*AT)(TA^{-1}T^*A)$$

$$=\{[A^{-1}(T^*{}_1 - iT^*{}_2)A](T_1 + iT_2)\}\{(T_1 + iT_2)[A^{-1}(T^*{}_1 - iT^*{}_2)A]\}$$

But $T^* = ATA^{-1}$. It then follows that

$$(T^{[*]}T)\big(TT^{[*]}\big)$$

$$=[A^{-1}(AT_1A^{-1} - iAT_2A^{-1})A](T_1 + iT_2)\}\{(T_1 + iT_2)[A^{-1}(AT_1A^{-1} - iAT_2A^{-1})A]$$

$=(T_1^2 + T_2^2)(T_1^2 + T_2^2)$. In like manner, $\big(TT^{[*]}\big)\big(T^{[*]}T\big) = (T_1^2 + T_2^2)(T_1^2 + T_2^2)$. Therefore T is A-binormal.

Remark 4.19: In view of the above theorem it can be deduced that if T is a decomposition such that $T = T_1 + iT_2$ where T_1 and T_2 are A-self-adjoint operators, then T is normal. (This follows from Theorem 3.13 and Corollary 3.14 [10]). A quick computation shows that $T^*T = (T^*{}_1 - iT^*{}_2)(T_1 + iT_2) = (AT_1A^{-1} - iAT_2A^{-1})(T_1 + iT_2)$ $= (T_1AA^{-1} - iT_2AA^{-1})(T_1 + iT_2) = (T_1 - iT_2)(T_1 + iT_2) = TT^*$.

Proposition 4.20 [10]: Every skew-adjont operator $T \in B(H)$ is binormal.

Proof: (See [10]).

Example 4.21: Define on the function Hilbert space $L^2[a\ b]$ a differential operator by $Tf = \frac{df}{dx}$ and show that it is skew-adjoint. Using integration by parts and the definition of an inner product space, it is clearly seen that

$\langle Tf, g\rangle = \int_a^b \frac{df}{dx}\overline{g(x)dx} = f(b)\overline{g(a)} - \int_a^b f(x)\overline{g'(x)}\,dx = -\int_a^b \frac{dg}{dx}f(x)dx = \langle f, -Tg\rangle$. This clearly shows that - $T^* = -T$ is a skew-adjoint operator.

Proposition 4.22: Every A-skew-adjoint operator $T \in B(H)$ is binormal.

Proof: Let T be A-skew-adjoint. Then $T^{[*]} = -T$, that is $T^* = -ATA^{-1}$. Here it is shown that $[T^*T, TT^*] = 0$ i.e. $(T^*T)(TT^*) = (TT^*)(T^*T)$.

$(T^*T)(TT^*) = ((-ATA^{-1}T)\big(T(-ATA^{-1})\big) = (ATA^{-1}T)(TATA^{-1})$. By [10, Corollary 4.3], T commutes with A and so $(T^*T)(TT^* = (TAA^{-1}T)(TTAA^{-1}) = (TT)(TT) = T^4$

$= (TT^*)(T^*T)$. We conclude that $T \in B(H)$ is binormal.

Corollary 4.23: Every A-skew-adjoint operator $T \in B(H)$ is A −binormal.

Proof: If T be A-skew-adjoint, then $T^{[*]} = -T$. A simple calculation shows that $[T^{[*]}T, TT^{[*]}] = 0$.

Remark 4.24: It is well known by earlier results that every skew-adjont operator T is A-skew-adjoint (see [10]). In view of this and the corollary above, it can also be deduced that every skew-adjoint operator $T \in B(H)$ is A −binormal.

5. Some Results on A-self adjoint and A-skew-adjoint Operators

In what follows, the relationship between A-self adjoint and A- skew -adjoint operators is investigated. It is known that every normal operator is quasinormal and every quasinormal operator is binormal. Using results in [10, Theorem 3.9] and [10, Proposition 4.4], some common behaviour of A-self adjoint and skew adjoint operators are established. It is also well known that every part of a skew adjoint is skew adjoint and so every part of a skew-adjoint operator is normal. Thus a skew adjoint operator has no completely non-normal part.

Proposition 5.1 [10]: Let T be an A-skew adjoint operator. Then T^n is A-self adjoint for even values of $n \in N$ and T^n is $(-A)$-skew adjoint for odd values of $n \in N$.

Remark 5.2: This proposition is simply interpreted as follows: that if T is A-skew adjoint, then T^n is $(-1)^n$ A-self adjoint. That is to say that T^n is A-skew- adjoint for odd values of $n \in N$ and T^n is A-self- adjoint for even values of n, which can also be extended to polynomials. The Lie Algebra $\mathbb{L}_A$ is closed under all odd degree polynomials over a field $\mathbb{F}$ while the Jordan Algebra $\mathbb{J}_A$ is closed under all polynomials over $\mathbb{F}$.

The following proposition now provides a characterization of an A-skew adjoint operator:

Proposition 5.3: Suppose $Q = TA$, where A is invertible and self-adjoint, then T is skew adjoint if and only if Q is A-skew adjoint.

Proof: Let T be skew-adjoint and $Q = TA$ with A invertible and self-adjoint. Then

$AQA^{-1} = ATAA^{-1} = AT = -AT = -Q^*$, that is Q is A-skew adjoint.

Conversely, let Q be A-skew adjoint with $Q = TA$, then $T = QA^{-1}$ and so

$T^* = A^{-1}Q^* = A^{-1}(-AQA^{-1}) = -QA^{-1} = -T$ that is T a skew-adjoint operator and this completes the proof.

Remark 5.4: The converse of the above proposition gives a more general property of an A-skew adjoint operator. It is also worth noting that the Lie Algebra $\mathbb{L}_A$ is a linear space but it is not closed under multiplication. Nonetheless, $\mathbb{L}_A$ is closed under the Lie bracket $[T_1, T_2] = T_1T_2 - T_2T_1$.

Question: Is there any relationship between the Lie Algebra $\mathbb{L}_A$ and the Jordan Algebra, $\mathbb{J}_A$? A possible answer to this question can be summarised in the following propositions:

Proposition 5.5: Let T_1 and T_2 be commuting A-skew adjoint linear operators. Then the product T_1T_2 is A-self-adjoint.

Proof: By definition T_1, $T_2 \in \mathbb{L}_A$ means that $T_1^* = -AT_1A^{-1}$ and $T_2^* = -AT_2A^{-1}$.

Since $[T_1, T_2] = 0$, we have $(T_1T_2)^* = T_2^*T_1^* = -AT_2A^{-1}(-AT_1A^{-1}) = AT_2T_1A^{-1} = AT_1T_2A^{-1}$. This proves that T_1T_2 is A-self-adjoint.

Proposition 5.6: Let T_1 be A-skew adjoint and T_2 be A-

self-adjoint. If T_1 and T_2 commute, then $T_1 T_2$ is A-skew-adjoint.

Proof: Given that T_1 A-skew adjoint and T_2 A-self-adjoint then
$(T_1 T_2)^* = T_2^* T_1^* = AT_2 A^{-1}(-AT_1 A^{-1}) = -AT_2 T_1 A^{-1}$. But T_1 and T_2 are commuting and so $-AT_2 T_1 A^{-1} = -AT_1 T_2 A^{-1}$. It follows that $(T_1 T_2)^* = -AT_1 T_2 A^{-1}$. This shows that $T_1 T_2$ is A-skew-adjoint as required.

Corollary 5.7: Let $T \in B(H)$ is an $A - skew\ adjoint$. Then

a) $\sigma_p(T) = \sigma_p(-T^*)$
b) $\sigma_c(T) = \sigma_c(-T^*)$
c) $\sigma_r(T) = \sigma_r(-T^*)$

Proof: Since T is $A - skew\ adjoint$ then by definition $T^* = -ATA^{-1}$. Thus, T and $-T^*$ are similar and hence have the same spectrum. As a consequence of this the above claims follow immediately since $\sigma(T)$ is the disjoint union of $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$.

Remark 5.8: From the above corollary, equality of spectra is evidently seen and this is indeed a necessary condition for A-skew-adjointness of an operator. As an example, consider the backward shift operator $T: l^2 \to l^2$ defined by $T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$ that is never A-skew-adjoint. Its adjoint (called the unilateral shift) is defined by $T^*(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. It is true (as an infinite matrix) that every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ (open unit disc centred at the origin) is in $\sigma_p(T)$ and that $\sigma_p(T^*) = \emptyset$. Also, $\{\lambda \in \mathbb{C}: |\lambda| < 1\} \subseteq \sigma_r(T^*)$. Hence T is not A-skew adjoint (for any A with the required properties: that it should self-adjoint and invertible) because the necessary condition for A-skew- adjointness is not satisfied (equality of spectra of T and T^*) i.e. $\sigma(T) \neq \sigma(T^*)$. (See a similar result on A-self-adjointness Corollary 3.8, pp 59 [2]).

This operator, namely, the backward shift operator $T: l^2 \to l^2$ is an example of an operator that is neither in the class of the Jordan algebra of A-self-adjoint nor the Lie algebra of the A-skew adjoint operators. However we should also note that there exist non-zero operators that are skew-adjoint and A-self-adjoint. This is illustrated in the example that follow:

Example 5.9: Let $T = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then a quick computation shows that T is both A-self-adjoint (that is $T^* = ATA^{-1}$) and skew-adjoint (i.e. $T^* = -T$). In view of this, it is seen that the only operator satisfying both conditions for A-self-adjointness and A-skew-adjointness is the zero operator.

6. Conclusion

From the preceding discussions and results above, it is clearly evident that A-self-adjoint, A-skew-adjoint and A-unitary operators are special casae of A-normal operators. It has also been noted that the class of A-self-adjoint operators contains some self-adjoint operators, some skew-adjoint operators and some which are neither of these categories. The backward shift operator as an example of such an operator as shown in the preceding section. That there exist operators which are skew-adjoint and A-self-adjoint but not A-skew-adjoint.

There is no class inclusion between A-self-adjoint and A-skew-adjoint operators. However, zero is the only operator that can satisfy this inclusion. The following class inclusions also hold:

Symmetry $\subsetneq$ Unitary $\subsetneq$ Normal $\subsetneq$ A- Normal and
Symmetry $\subsetneq$ Self- adjoint $\subsetneq$ Normal $\subsetneq$ A- Normal
Skew –adjoint $\subsetneq$ Normal $\subsetneq$ A- Normal. In addition the intersection of the class of self-adjoint and unitary operators yields a symmetry, i.e.

$$\{Self - adjoint\} \cap \{Unitary\} = \{Symmetry\}.$$

Just like A-self-adjoint operators, the spectrum of an A-skew-adjoint operator $-T$ and the adjoint operator T^* is equivalent, that is $\sigma(T) = \sigma(-T^*)$.

Finally, it has also been established that A-almost similarity is an equivalence relation just like other equivalences like unitary and almost similarity on a Hilbert space. A-almost similar operators have equal spectra if they are projection operators.

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