

On the Accelerated Overrelaxation Method

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Abstract: A new variant of the accelerated over relaxation (AOR) method for solving systems of linear algebraic equations, the KAOR method is established. The treatment depends on the use of the extrapolation techniques introduced by Hadjidimos (1978) and the KSOR introduced by Youssef (2012) and its modified version published in (2013). The KAOR is an extrapolation version of the KSOR. This approach has enables us in establishing eigenvalue functional relations concerning the eigenvalues of the iteration matrices and their spectral radii. Moreover, our eigenvalue functional relation depends directly on the eigenvalues of Jacobi iteration matrix. The proposed method considers the advantages of the AOR in addition to those of the KSOR. A graphical representation of the behavior of the spectral radius near the minimum value illustrates the smoothness in the choice of optimum parameters.

Keywords: SOR, KSOR, AOR, QAOR, KAOR

1. Introduction

Consider a linear system of algebraic equations of the form

$$AX = b \quad (1)$$

$A \in R^{n \times n}$ with non-vanishing diagonal elements. It is generally, accepted that the matrix of coefficients can be written in the form

$$A = D - A_L - A_U \quad (2)$$

Where D is the diagonal part of the matrix A and A_L and A_U are the strictly lower and upper triangular parts of the matrix A , respectively. Hadjidimos, [1] in 1978 introduced the accelerated overrelaxation iterative method (AOR) as a two parameter generalization of the standard Jacobi, Gauss-seidel, and successive overrelaxation (SOR) methods, [2, 3] as shown figure (1),

It is common to normalize the system (1) such that the diagonal elements are all equal one in this case the matrix of coefficients A is written as

$$A = I - L - U$$

Accordingly, the AOR method can be written in the form

$$X^{[n+1]} = L_{r, \omega} X^{[n]} + \omega(I - rL)^{-1} b$$

$$L_{r, \omega} = (I - rL)^{-1} [(1 - \omega)I + (\omega - r)L + \omega U]$$

Moreover, the (AOR) can be seen as an extrapolation of (SOR) method with overrelaxation parameter r and extrapolation parameter $s = \omega/r$, [4, 5].

Youssef [6] in 2012, introduced the KSOR as a new variant of the SOR, Youssef and Taha [7] in 2013 introduced the modified versions of the KSOR, the main idea of the KSOR is the assumption of using the current component in the updating process instead of using the most recent calculated values in the SOR methods. The KAOR method is introduced as a new variant of the AOR method. The KAOR can be introduced through two equivalent treatment of the KSOR approach the first is the interpolation process in terms of the sub-matrices as in Hadjidimos, [1] and the second is the extrapolation of the KSOR method. The second approach confirms the first and moreover facilitates the treatment in establishing eigenvalue functional relation, [4, 5].

2. The KAOR Method

For the numerical solution of Eq. (1) the most general linear stationary iterative scheme whose matrix of coefficients are linear functions of the components of the matrix A and

the coefficient of the new iterate is an at most lower triangular matrix is

$$(a_1D + a_2A_L) X^{n+1} = (a_3D + a_4A_L + a_5A_U) X^n + a_6b, \quad n=0, 1, \dots \quad (3)$$

as in Hadjedemos [1],

where $a_i, i = 1(1)6$ are constants to be determined. Sufficient conditions for scheme (3) to be consistent with Eq. (1) are

$$(a_1 - a_3)D + (a_2 - a_4)A_L - a_5A_U = a_6A, \quad a_6 \neq 0 \quad (4)$$

In view of (2), the first relation of (4) gives

$$a_1 - a_3 = a_6, \quad a_2 - a_4 = -a_6 \quad \text{and} \quad -a_5 = -a_6$$

The above set of equations has the following two-parameter solution

$$a_1 = k+1, \quad a_2 = -k, \quad a_3 = 1+k-\omega, \quad a_4 = \omega-k \quad \text{and} \quad a_5 = a_6 = \omega,$$

where k and $\omega \neq 0$ are fixed parameters.

Consequently, (3) becomes

$$((k+1)I - kL)x^{(n+1)} = [(k-\omega)I + (\omega-k)L + \omega U]x^n + \omega C, \quad n=0, 1, \dots \quad (5)$$

where we have

$$L = D^{-1}A_L, \quad U = D^{-1}A_U, \quad C = D^{-1}b \quad (6)$$

and I the identity matrix of order N . The form (5) is the KAOR method or $M_{k,\omega}$ -method. We observe that for specific values of the parameters k and ω , the $M_{k,\omega}$ -method reduces to well-known methods. Thus

$M_{0,1}$ -method is the Jacobi method,

$M_{0,\omega}$ -method is the extrapolated Jacobi method,

$M_{\omega,\omega}$ -method is the KSOR method.

Now we will use the notations $L_{k,\omega}$ for the iteration matrix of the method in (5),

$$L_{k,\omega} = ((k+1)I - kL)^{-1} [(1+k-\omega)I + (\omega-k)L + \omega U]$$

$\rho(L_{k,\omega})$ is the spectral radius of $L_{k,\omega}$.

It should be noted that, except for the case $\omega = 0$, the KAOR method is essentially the Extrapolated (E)KSOR method with overrelaxation parameter k and extrapolation one $s = \omega/k$, for it is easy to show that

$$L_{k,\omega} = sL_{k,k} + (1-s)I$$

Thus if v is an eigenvalue of $L_{k,k}$ ($k \neq 0$) and λ , the corresponding one of $L_{k,\omega}$, we have that

$$\lambda = sv + (1-s) \quad (7)$$

By considering the KAOR method as an extrapolation of the KSOR method, we can use the extrapolation theorem in the evaluations of the possible values of the parameters k and ω for the KAOR method.

Extrapolation Theorem

If the scheme $[x^{n+1} = Tx^n + c]$ converges $[\rho(T) < 1]$ and $0 < \omega < 2/[1 + \rho(T)]$, then the extrapolated scheme $[x^{n+1} = [(1-\omega)I + \omega T]x^n + \omega C]$ converges.

Proof, [4].

Theorem 1: the domain of convergence of the relaxation parameter of the KSOR method can be written in the form $\omega \in R - [-2, 0]$, [6].

Theorem 2: the sufficient conditions for KAOR to converge

- 1) for $k \in]0, \infty[$, is $0 < \omega < 2k/[1 + \rho(L_{k,k})]$
- 2) for $k \in]-\infty, -2[$, is $2k/[1 + \rho(L_{k,k})] < \omega < 0$

Proof, (1) for $k \in]0, \infty[$, from Theorem 1 the KSOR converges, and from Extrapolation Theorem the KAOR method converges for $0 < \omega/k < 2/[1 + \rho(L_{k,k})]$ where (ω/k) is the extra-polation parameter, and where k is positive then the condition is $0 < \omega < 2k/[1 + \rho(L_{k,k})]$.

(2) for $k \in]-\infty, -2[$, by the same way of (1), but where k is negative and the condition will be

$$2k/[1 + \rho(L_{k,k})] < \omega < 0.$$

Here we can say that the KAOR method extends the range for choosing the parameters k and ω which reduce the sensitive of the spectral radius of the iteration matrix with the changes in k and ω like the KSOR method, as shown in figure (2).

The discussion of the AOR for general linear systems is very complicated so the publications have considered the practical cases, [8, 9, 10] like irreducible matrices with weak diagonal dominance, L matrices and the consistently ordered matrices.

In the next section we will discuss, under different assumptions on the original matrix A , the conditions on the parameters k and ω so that the $M_{k,\omega}$ -method converges.

Proof. It is obvious that if the $M_{k,\omega}$ -method converges so does the $M_{0,1}$ -method. Assume now that $\lambda = \rho(L_{k,\omega}) > 1$. Because of our assumptions we easily get that

$$[(k+1)I - kL]^{-1} = \frac{1}{(1+k)} [I + (\frac{k}{k+1})L + (\frac{k}{k+1})^2 L^2 + \dots + (\frac{k}{k+1})^{n-1} L^{n-1}] \geq 0$$

thus, for the iteration matrix we have that

$$L_{k,\omega} = ((k+1)I - kL)^{-1} [(1+k-\omega)I + (\omega-k)L + \omega U] \geq 0$$

Since $L_{k,\omega}$ is a nonnegative matrix, λ is an eigenvalue of $L_{k,\omega}$. If $V \neq 0$ is the corresponding eigenvector, we have $L_{k,\omega} V = \lambda V$ from which we can obtain that

$$\frac{(\omega + k(\lambda - 1))L + \omega U}{\omega} V = \frac{(1+k)(\lambda - 1) + \omega}{\omega} V$$

This implies that $\frac{(1+k)(\lambda - 1) + \omega}{\omega}$ is an eigenvalue of $(\frac{\omega + k(\lambda - 1)}{\omega} L + U)$. Therefore

$$\frac{(1+k)(\lambda - 1) + \omega}{\omega} \leq \rho(\frac{\omega + k(\lambda - 1)}{\omega} L + U)$$

It is obvious that $\frac{\omega + k(\lambda - 1)}{\omega} \geq 1$ so that

$$0 \leq \frac{\omega + k(\lambda - 1)}{\omega} L + U \leq \frac{\omega + k(\lambda - 1)}{\omega} (L + U) = \frac{\omega + k(\lambda - 1)}{\omega} L_{0,1}$$

Then

$$(1+k)(\lambda - 1) + \omega \leq (\omega + k(\lambda - 1))\rho(L_{0,1})$$

from which we can easily obtain that $\rho(L_{1,0}) \geq 1$. Since we have proved that if $\lambda \geq 1$, then $\rho(L_{1,0}) > 1$, we can readily obtain that $\rho(L_{1,0}) < 1$ implies $\lambda < 1$ so that if the $M_{1,0}$ -method converges then so does the $M_{k,\omega}$ -method.

3.3. Consistently Ordered Matrices

In this section we assume that matrix A is a consistently ordered one, that is, a matrix for which the expression $\det(aA_L + a^{-1}A_U - bD)$ is independent of a for $a \neq 0$ and for all b . As can be easily found out, the analysis of this section also applies in the case where A is a matrix which has property A. LEMMA. If A is a consistently ordered matrix with non-vanishing diagonal elements, and if $\mu \neq 0$ is an eigenvalue of $L_{1,0}$ of multiplicity p , then $-\mu$ is also an eigenvalue of $L_{1,0}$ of multiplicity p .

Proof. See Theorems 3.4 and 2.2 on pages 147 and 142, respectively, of Young [2].

THEOREM. *If A is a consistently ordered matrix with non-vanishing diagonal elements, and if μ is an eigenvalue of $L_{1,0}$ and v satisfies*

$$(v + v\omega - 1)^2 = v\mu^2 \omega^2 \tag{15}$$

Then v is an eigenvalue of $L_{\omega,\omega}$ and vice versa.

Proof. See [6]

$$[(1+k-\omega)I + (\omega-k)L + \omega U] \geq 0$$

and also that

THEOREM. If A is a consistently ordered matrix with non-vanishing diagonal elements, and if μ is an eigenvalue of $L_{1,0}$ and λ satisfies

$$[(k+1)\lambda - (1+k-\omega)]^2 = \mu^2 \omega [k(\lambda - 1) + \omega] \tag{16}$$

then λ is an eigenvalue of $L_{k,\omega}$ and vice versa.

Proof. Since the requirements of theorem 1 are fulfilled, for $\omega \neq 0$, we substitute the value of v in terms of λ from (7) into (15) and its equivalent relationship (16) follows. For $\omega = 0$ it is easy to show that, λ satisfies (16) and vice versa.

THEOREM. If A is a consistently ordered matrix with non-vanishing diagonal elements, and if $L_{1,0}$ has real eigenvalues $\mu_i, |i|=1(1)N$ such that, $0 < \min_i |\mu_i| = \max_i |\mu_i| < 1$,

then for $(k,\omega) = (\frac{-(1+(1-\mu^2)^{1/2})(-2-2(1-\mu^2)^{1/2}+\mu^2)}{\mu^2(-1-(1-\mu^2)^{1/2}+\mu^2)}, \frac{-2(1+(1-\mu^2)^{1/2})}{\mu^2})$

Or $(\frac{-(\mu^2-2+2(1-\mu^2)^{1/2})^2}{\mu^2(-2(1-\mu^2)^{1/2}+\mu^2(1-\mu^2)^{1/2}-2\mu^2+2)}, \frac{-2}{(1+(1-\mu^2)^{1/2})})$,

$\rho(L_{k,\omega}) = 0$.

Proof. From the Lemma, μ will be an eigenvalue of $L_{0,1}$. Since μ^2 assumes one and only one fixed value we can derive values for $\omega \neq 0$ so that (15) has a double root.

These values are

$$\omega_1 = \frac{-2(1+(1-\mu^2)^{1/2})}{\mu^2}, \omega_2 = \frac{-2}{(1+(1-\mu^2)^{1/2})} \tag{17}$$

and the double root for v will be given by

$$v = \frac{2(1+\omega) + \mu^2 \omega^2}{2(1+\omega)^2} \tag{18}$$

Since v has only one value it is easy to determine s (i.e. k) from (1.7) so that $\lambda = 0$. For this we must have

$$k = \omega/(1-v) \tag{19}$$

Thus we finally obtain

$$k_1 = \frac{-(1+(1-\mu^2)^{1/2})(-2-2(1-\mu^2)^{1/2}+\mu^2)}{\mu^2(-1-(1-\mu^2)^{1/2}+\mu^2)}$$

$$k_2 = \frac{-(\mu^2-2+2(1-\mu^2)^{1/2})^2}{\mu^2(-2(1-\mu^2)^{1/2}+\mu^2(1-\mu^2)^{1/2}-2\mu^2+2)} \tag{20}$$

The pairs (k_1, ω_1) and (k_2, ω_2) give $\rho(L_{k,\omega}) = 0$.

4. Numerical Examples

Three examples from recent publications to illustrate that the KAOR preserves the properties of the AOR, the

sensitivity of the relaxation parameter is considerably relaxed in the KAOR in comparison with the AOR as in the KSOR in comparison with the SOR [6], figure (2) below illustrates this behavior. The range of the relaxation parameters is extended as in the KSOR (R-[-2, 0]) as shown in table 3, which is ignored in the so called QAOR [11].

Example 1 consider the matrix [8]

$$\begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & \frac{-18}{5} & 6 \\ \frac{24}{5} & \frac{1}{5} & 1 & 0 \\ \frac{12}{5} & \frac{1}{5} & 0 & 1 \end{bmatrix}$$

the eigenvalues of the Jacobi matrix are $\pm\sqrt{24}/5$, each one with multiplicity two. Thus $0 < \mu = \sqrt{24}/5 = \bar{\mu} < 1$ Since the restrictions of Theorem 3 are satisfied, the corresponding values for the KAOR method are $(k_1, \omega_1) = (-15/2, -5/2)$ and $(k_2, \omega_2) = (10/3, -5/3)$, where both pairs give $\rho(L_{k,\omega}) = 0$, as it was in the AOR.

Example 2 Consider the system [6]

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Table 1. The behavior of the spectral radius $\rho(KAOR)$ near the minimum value.

$\rho(KAOR)$	k
0.0719	-14.9300
0.0719	-14.9290
0.0718	-14.9280
0.0719	-14.9270
0.0719	-14.9260
0.0720	-14.9250

Table 2. the behavior of the spectral radius of KAOR near the minimum value.

$\rho(AOR)$	r
0.0927	1.0700
0.0858	1.0710
0.0722	1.0720
0.0740	1.0730
0.0758	1.0740
0.0775	1.0750

The behavior of the spectral radius of the KAOR iteration matrix is very smooth than the behavior of spectral radius of the AOR iteration matrix around the optimum eigenvalue see figure (2).

Example 3 Consider the system [11]

$$A = \begin{cases} a_{ij} = \frac{1}{10j} - \frac{1}{20}, i > j \\ a_{ii} = 1, \\ a_{ij} = \frac{1}{10(i-j)} - \frac{1}{20}, i < j \end{cases}$$

Table 3. $\rho(KSOR)$ and $\rho(KAOR)$.

n	$\omega(KSOR)$	k	$\omega(KAOR)$	$\rho(KSOR)$	$\rho(KAOR)$
8	-17.85	-16.3	-16.2	0.13537756	0.13537172
16	-7.82	-11.9	-12.5	0.240810	0.240495
20	-3.6	-3	-2.5	0.49703	0.39082

As we see in the table 3: where n is the size of A , the spectral radius of the KAOR iterative matrix is less than spectral radius of the KSOR iterative matrix and less than the spectral radius of the QAOR iterative matrix introduced in [11], that is to say, the KAOR iteration is faster than the QAOR under the same conditions.

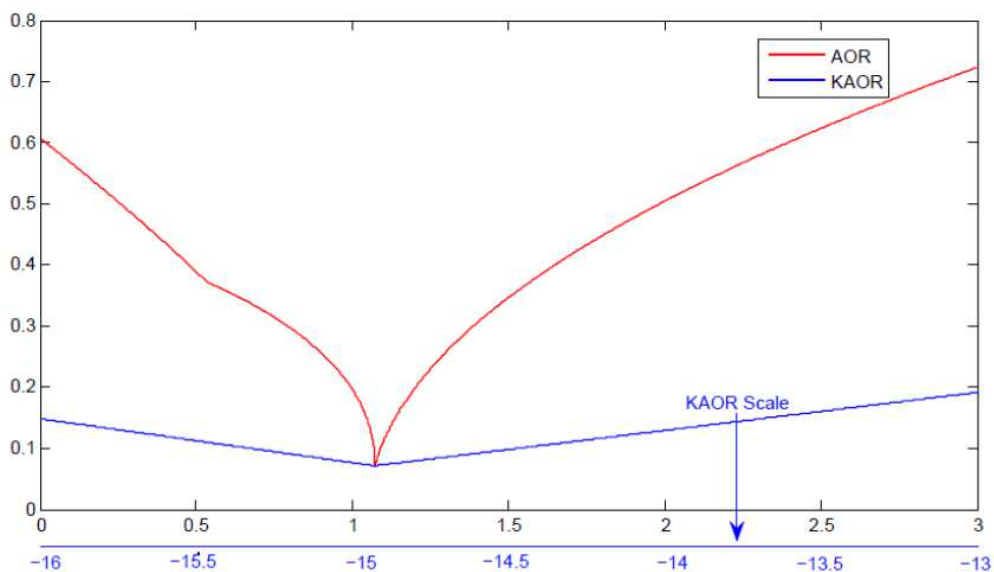


Fig. 2. The behavior of the spectral radius of L_{AOR} and L_{KAOR} near the minimum value.

5. Conclusion

The KAOR is an extrapolation version of the KSOR similar to the situation in the AOR with the SOR. As in the AOR, the KAOR reduces to well-known methods at specific values of the parameters κ , ω like the Jacobi, extrapolated Jacobi and the KSOR. In comparison with the AOR the KAOR preserves the properties of the KSOR, the behavior of spectral radius is relaxed near the optimum values, figure 2.

It is noticed that some recent publication [11] has ignored the negative values of the relaxation parameters in spite of their high importance as we have seen the optimum values occurs in this part of the domain.

Shi-liang Wu, Yu-Jaun Liu [11], introduced the QAOR as a generalization of the KSOR from the AOR point of view, their relational graph figure 1 illustrates that The QAOR is a generalization of the KSOR ($\omega=r$) but it is not an extrapolation version as the situation in the AOR and the SOR. Moreover, as shown in the relational graph the KAOR generalizes, the AOR ($k = \frac{r}{1-r}$, $\omega = \frac{\omega}{1-r}$), and generalizes the KSOR ($\omega = k$), generalizes the Jacobi ($\omega = 1$, $k = 0$) and the extrapolated Jacobi ($k = 0$).

References

- [1] A. Hadjidimos, Accelerated Overrelaxation Method, Math. Comput. , 32, pp149-157, 1978.
- [2] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.
- [3] R.S. Varga, Matrix Iterative Analysis, prentice-Hall, Englewood cliffs, N. J., 1962.
- [4] A. Hadjidimos, A. Yeyios, The Principale of Extrapolation in Connection With the Accelerated Overrelaxation Method, Linear Algebra and its Applications 30, 115-128 , 1980.
- [5] N. M. Missirlis, D. J. Evans, The Extrapolated Successive Overrelaxation (ESOR) Method for Consistently Ordered Matrices, Internat. J. Math. & Math. sci. Vol. 7, No. 2, pp. 361-370, 1984.
- [6] I.K. Youssef, On the Successive Overrelaxation method, J. Math. Stat., 8 pp.176-184, 2012.
- [7] I.K. Youssef, A.A. Taha, On the Modified Successive Overrelaxation Method, Appl. Math.Comput., 219, pp. 4601-4613, 2013.
- [8] G. Avdelas and A. Hadjidimos, Optimum Accelerated Overrelaxation Method in a Special Case, Mathematics of computation Vol. 36, No. 153, 1981.
- [9] M. Madalena Martins, On an Accelerated Over relaxation Iterative Method for Linear Systems With Strictly Diagonally Dominant Matrix, Mathematics of Computation, Vol. 35, No. 152, pp. 1269-1273, 1980.
- [10] A.J. Hughes Hallet, The Convergence of Accelerated Overrelaxation Iterations, Mathematics of Computation, Vol. 47, No. 175, pp. 219-223, 1986.
- [11] Shi-liang Wu, Yu-Jun Liu, A new Version of the Accelerated Overrelaxation Iterative Method, Journal of Applied Mathematics, 2014.