Pure and Applied Mathematics Journal
2015; 4(1): 24-25
Published online January 30, 2015 (http://www.sciencepublishinggroup.com/j/pamj)
doi: 10.11648/j.pamj.20150401.13
ISSN: 2326-9790 (Print); ISSN: 2326-9812 (Online)

Certain ranks of the quotient semigroup *S/ρ* and prime subsets of a semigroup

Melis Minisker

Mustafa Kemal University, Department of Mathematics, Antakya, Hatay, Turkey

Email address:
minisker38@gmail.com

To cite this article:
Melis Minisker. Certain Ranks of the Quotient Semigroup *S/ρ* and Prime Subsets of a Semigroup. *Pure and Applied Mathematics Journal.* Vol. 4, No. 1, 2015, pp. 24-25. doi: 10.11648/j.pamj.20150401.13

Abstract: In this paper we find a relationship between $r2(S)$ and $r2(S/\rho)$ where S is a finite semigroup, ρ is a congruence on S and S/ρ is the quotient semigroup $(|S| \geq 2$ and $|S/\rho| \geq 2)$. We also determine $r5(S/\rho)$ under certain conditions. Moreover we find prime subsets of S/ρ.

Keywords: Rank, Prime Subset, Congruence, Independent Set

1. Introduction

In algebra many studies have been constructed based upon the notion "rank". Problems about rank of a group, rank of an algebra and rank of a semigroup have been widely studied in many articles for years.

In *[3]* the notion of rank of a semigroup and ranks of some certain semigroups are determined. In *[1]* and *[2]* the small rank, lower rank, intermediate rank, upper rank and large rank(i.e. $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$) are defined. In the following definitions these ranks are given:

$r_1(S)=max\ \{k:$ every subset of cardinality k is independent$\}$

$r_2(S)=min\{k:$ there exists a subset U of cardinality k such that U generates $S\}$

$r_3(S)=max\ \{k:$there exists a subset U of S of cardinality k which is independent and which generates $S\}$

$r_4(S)=max\ \{k:$there exists a subset U of S of cardinality k which is independent$\}$

$r_5(S)=min\ \{k:$every subset U of cardinality k generates $S\}$

One can see that $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$. Let S be a finite semigroup. Let $U \subseteq S$. If for all $x \in U$, $x \notin \langle U / \{x\} \rangle$ then U is an independent set. In *[4]* ranks of certain semigroups are determined.

If $x, y \in S$ and $(s, t) \in \rho$ and $(xs, yt) \in \rho$ then ρ is a congruence on S. If ρ is a congruence on S then ρ is a subsemigroup of $S \times S$. $S/\rho=\{x\rho: x \in S\}$. If ρ is a congruence on S then S/ρ is a semigroup with the operation $(x\rho).(y\rho)=(xy)\rho$. S/ρ is the quotient semigroup.

In *[5]* it is shown that if $T=S$ or $T=S/\rho$ then $r_i(T) \leq r_i(\rho)$ for $i=2, 3, 4, 5$. In general it is not equal.

In this paper in Section 2 we determine the relationship between $r_2(S)$ and $r_2(S/\rho)$. We also obtain some results about $r_5(S)$ and $r_5(S/\rho)$.

Let S be a finite multiplicative semigroup. Let $U \subseteq S$. If for all $a, b \in S$, $ab \in U$ implies $a \in U$ or $b \in U$ then U is called a prime subset of S. In Section 3 we obtain some results about the prime subsets of the quotient semigroup S/ρ. We also determine the union and intersection of two prime subsets of a finite semigroup S.

2. Some Ranks of Quotient Semigroup

We find a relationship between $r_2(S)$ and $r_2(S/\rho)$ where S is a finite semigroup and ρ is a congruence on S $(|S| \geq 2$ and $|S/\rho| \geq 2)$. We also show that $r_5(\frac{S}{\rho}) = \left|\frac{S}{\rho}\right|$ when $r_5(S) = |S|$.

Lemma 2.1: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S. Assume that $|S| \geq 2$ and $|S/\rho| \geq 2$. Then $r_2(S/\rho) \leq r_2(S)$.

Proof: Let S be a finite semigroup and $|S| \geq 2$. Let ρ be a congruence on S. Let $|\rho| \geq 2$ and $|S/\rho| \geq 2$. Let $r_2(S)=m$. Then S has a minimal generating set $A=\{x_1, x_2, \ldots, x_m\}$. Assume that $r_2(S/\rho)=m'$. Assume that $A'=\{x_1\rho, x_2\rho, \ldots, x_m\rho\}$. Let $s\rho \in S/\rho$. Since $s \in S$ and S is a generating set $s= x_{i_1}x_{i_2}\ldots x_{i_k}$ $(1 \leq i_1, i_2, \ldots i_k \leq m)$. Then $s\rho= (x_{i_1}\rho)(x_{i_2}\rho)\ldots(x_{i_k}\rho)$. Thus $A'=\{x_1\rho, x_2\rho, \ldots, x_m\rho\}$ is a generating set for S / ρ. Since $r_2(S/\rho)=m'$ we have $m' \leq m$.

By the help of the next theorem we give a relationship between irreducible elements of S and irreducible elements of S/ρ.

Theorem 2.2: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S. Assume that $|S|\geq 2$ and $|S/\rho|\geq 2$. If there is an irreducible element in S, there is also an irreducible element in S/ρ.

Proof: Let $x\in S$ be an irreducible element. We determine whether $x\rho\in S/\rho$ is irreducible or not. Let $x\rho= y\rho.\ z\rho$. Let $y\rho=\{y_1, y_2, ..., y_n\}$ and $z\rho=\{z_1, z_2, ..., z_m\}$. Let $y\rho.\ z\rho= yz\rho=\{ y_1 z_1, y_1 z_2, ..., y_1z_m, y_2 z_1, y_2 z_2, ..., y_2z_m, ..., y_nz_m\}$. Since $x\rho= yz\rho$, $x\rho=yz\rho$ and $x\in x\rho$ there is $\exists k, l$ such that $x= y_kz_l$ $(1\leq k\leq n, 1\leq l\leq m)$. Since x is an irreducible element we have $x=y_k$ or $x=z_l$. Thus $x\in y\rho$ or $x\in z\rho$. If $x\in y\rho$ then since $x\in x\rho$ we have $x\rho\cap y\rho\neq\varnothing$. We obtain $x\rho= y\rho$. By the same way, if $x\in z\rho$ and $x\in x\rho$ we have $x\rho\cap z\rho\neq\varnothing$. Then $x\rho= z\rho$. As a result $x\rho= y\rho$ or $x\rho=z\rho$. As a result $x\rho\in S/\rho$ is an irreducible element.

In the following Theorem we give the relationship between $r_5(S)$ and an irreducible element of a finite semigroup.

Theorem 2.3: (see[2],Corollary 4) Let S be a finite semigroup. S contains an irreducible element if and only if $r_5(S)=|S|$.

Lemma 2.4: Let S be a finite semigroup and S/ρ be the quotient semigroup where ρ is a congruence on S. Assume that $|S|\geq 2$ and $|S/\rho|\geq 2$. If $r_5(S)= |S|$ then $r_5(S/\rho)=|S/\rho|$.

Proof: By the previous theorem we have S has an irreducible element. So $r_5(S)=|S|$. (*see [2]*). Then by the previous theorem S/ρ has an irreducible element. We obtain

$$r_5(\frac{S}{\rho})=\left|\frac{S}{\rho}\right|.$$

3. Prime Subsets of a Semigroup

In [6] Kumar J. and Krishna, K.V. have defined the prime subsets of a semigroup as follows.

Definition 3.1: Let S be a finite multiplicative semigroup. Let $U\subseteq S$. If for all $a, b\in S$, $ab\in U$ implies $a\in U$ or $b\in U$ then U is called a prime subset of S.

In the following Theorem we determine the prime subsets of S/ρ.

Theorem 3.2: Let $U\subseteq S$ be a prime subset of S. Let $U'=\{u\rho: u\in U\}\subseteq S/\rho$. Then U' is a prime subset of S/ρ.

*Proof:*Let $U\subseteq S$ be a prime subset of S. Then for all $a, b\in S$, $ab\in U$ implies $a\in U$ or $b\in U$. Let $x\rho=\{ s\in S: x\rho S\}$ $(x\in S)$. Let $S/\rho=\{ x\rho: x\in S\}$ and $U'=\{u\rho: u\in U\}$. Assume that for $x\rho$, $y\rho\in S/\rho$, $x\rho.\ y\rho\in U'$. Then $x\rho.y\rho=xy\rho\in U'$. Thus there is some $u\in U$ such that $xy=u$. Since U is a prime subset of S we have $x\in U$ or $y\in U$. We obtain $x\rho\in U'$ or $y\rho\in U'$. So U' is a prime subset of S/ρ.

Corollary 3.3: For every prime subset $U\subseteq S$, there exists a prime subset $U'\subseteq S/\rho$.

We try to determine the union of prime subsets of a finite semigroup S. In the next theorem we examine the union of prime subsets of S.

Lemma 3.4: Let S be a finite semigroup and U_1 and U_2 be prime subsets of S. Then $U_1\cup U_2$ is a prime subset of S.

Proof: Let $xy\in U_1\cup U_2$. Then xy $\in U_1$ or xy $\in U_2$. Since U_1 is a prime subset of S then x$\in U_1$ or y$\in U_1$. In each cases $x\in U_1\cup U_2$ or $y\in U_1\cup U_2$. We obtain $U_1\cup U_2$ is a prime subset of S.

We also determine the intersection of prime subsets of S in the following Lemma.

Lemma 3.5: Let S be a finite semigroup and U_1 and U_2 be prime subsets of S. Then $U_1\cap U_2$ is a prime subset of S.

Proof: Let $xy\in U_1\cap U_2$. Then xy$\in U_1$ and xy$\in U_2$. Since U_1 is a prime subset of S then $x\in U_1$ and $y\in U_1$. Also U_2 is a prime subset of S. Then we have $x\in U_2$ and $y\in U_2$. We obtain $x\in U_1\cap U_2$ and $y\in U_1\cap U_2$. So $U_1\cap U_2$ is a prime subset of S.

4. Conclusion

In [5] Theorem 1 it is shown that $r_1(S)=r_1(\rho)=r_1(S/\rho)$ when ρ is not a royal semigroup. $((|S|\geq 2$and$|S/\rho|\geq 2)$ If ρ is a royal semigroup then $r_1(S)=|S|$ and $r_1(S/\rho)=|S/\rho|$. In Section 2 we show that $r_2(S/\rho)\leq r_2(S)$. Moreover we show that $r_5(\frac{S}{\rho})=\left|\frac{S}{\rho}\right|$ when $r_5(S)=|S|$.

Let $U\subseteq S$ be a prime subset of S. In Section 3 we determine $U'\subseteq S/\rho$ which is a prime subset of S/ρ. As a result for each prime subset of S one can find a prime subset of the quotient semigroup. So each prime subset of a semigroup S coincides with a prime subset of S/ρ.

Moreover we show that union of two prime subsets of a finite semigroup S and intersection of two prime subsets is also a prime subset of S.

Acknowledgement

The author would like to thank to Asistant Prof. Dr. Basri Caliskan who has helped the arisement of the problem of determining certain ranks of the quotient semigroup S/ρ.

References

[1] John Macintosh H. and Maria Isabel R. "Rank Properties in Finite Semigroups", Communications in Algebra, Vol 27 (11), pp. 5333-5347, 1999.

[2] John Macintosh H. and Maria Isabel R. "Rank Properties in Finite Semigroups II: The Small Rank and The Large Rank", Southeast Asian Bulletin of Math.,Vol24, pp. 231-237, 2000.

[3] John Macintosh H. Foundations of SemigroupTheory, Clarendon Press, Oxford, 1995.

[4] Melis M. "Ranks of Certain Semigroups", Semigroup Forum, Vol 28, pp. 99-105, 2009.

[5] Fikret K. "Relations Between Ranks of Certain Semigroups", Selçuk J. Of Applied Math., Vol. 12, No 1, pp. 123-126, 2011.

[6] Jitender K. and Krishna K.V. "The Large Rank of a Finite Semigroup Using Prime Subsets", Semigroup Forum, DOI 10.1007/s00233-014-9577-0.