

Stability of a Jensen's functional equation in Banach spaces

Seyed Habib Shakoory

Electrical and Computer Engineering Department, Seraj Higher Education Institute, Tabriz, Iran

Email address:

Habibshakoory@yahoo.com

To cite this article:

Seyed Habib Shakoory. Stability of a Jensen's Functional Equation in Banach Spaces. *Pure and Applied Mathematics Journal*. Vol. 3, No. 4, 2014, pp. 92-94. doi: 10.11648/j.pamj.20140304.13

Abstract: Our aim in this note is to study the Hyers-Ulam stability and we investigate the Hyers-Ulam stability of generalized Jensen's Functional Equation in Banach Spaces: $f((x + y)/2) = (f(x) + f(y))/2$

Keywords: Hyers-Ulam Stability, Jensen's Functional Equation, Approximation Theory

1. Introduction

The Stability of Functional Equations seem to have been first studied by Ulam [5] in 1940 and solved in the next year for the Cauchy Functional Equations by Hyers [2]. In the following year, Hyers theorem gave a partial affirmative answer to the question of Ulam-Hyers theorem and it was generalized by Aoki [1]. For additive mapping by Rassias [4] and for linear mapping by considering an unbounded Cauchy difference during the past decades, several stability problems for various Functional equations have been investigated by a number of mathematicians, and we refer the reader to [cf. Gajda 1991, Gavruta 1994, Vaezi and park 2008, vaezi and zamani 2009, vaezi and dehgani 2010, Vaezi and Shakoory 2012, shakoory 2013] and also to the references cited therein. Another famous equation that is closely related to the Cauchy functional equation is Jensen's functional equation:

$$f((x + y)/2) = (f(x) + f(y))/2$$

The first result on the stability of the classical Jensen's functional equation was given by Kominek [3]. In this paper, we investigate the Hyers-Ulam stability of the Jensen's functional equation for linear bounded mapping from Banach spaces into Banach spaces.

2. Main Results

2.1. Definition

Let X be a set. A function $d: X \times X \rightarrow [0, \infty)$ is called a

generalized metric on X if d satisfies:

- (1) $d(x, y) = 0$ if and only if $x = y$.
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$.
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

2.2. Definition

A (real) complex normed space is a (real) complex vector space X together with a map: $X \rightarrow \mathbb{R}$, called the norm and denoted $\| \cdot \|$, such that

- (1) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.
- (2) $\|\alpha x\| = |\alpha| \|x\|$, for all $x \in X$ and all $\alpha \in \mathbb{R}$ or (\mathbb{C}) .
- (3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

2.3. Definition

A complete normed space is called a Banach space. Thus, a normed space X is Banach space if every Cauchy sequence in X converges (where X is given the metric space structure as outlined above).

2.4. Definition

Let W be a Banach space. f a differentiable map on V into W so that

$$\|f((x + y)/2) - f(x/2) - f(y/2)\| \leq \epsilon$$

for all $x, y \in V$, we say that the Hyers-Ulam stability holds for f , if there exist a $K > 0$ and a differentiable map T on V into W such that

$$\|T((x+y)/2) - T(x/2) - T(y/2)\|_B \leq \varepsilon$$

and

$$\|f(x) - T(x)\|_B \leq K \varepsilon$$

holds for every $x, y \in V$, we call such K a Hyers-Ulam stability constant for Jensen's functional equation:

$$f((x+y)/2) = (f(x) + f(y))/2$$

Now, the main result of this work is given in the following theorem.

2.4.1. Theorem

Let V and W be Banach spaces. A mapping $f: V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation:

$$2^n f((x_1 + x_2 + \dots + x_{2^n}) / (2^n)) = \sum_{i=1}^{2^n} f(x_i)$$

for all $x_1, x_2, \dots, x_{2^n} \in V$ if and only if the mapping $f: V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof:

Assume that a mapping $f: V \rightarrow W$ satisfies

$$2^n f((x_1 + x_2 + \dots + x_{2^n}) / (2^n)) = \sum_{i=1}^{2^n} f(x_i)$$

Put

$$x_1 + x_2 + \dots + x_{2^{n-1}} = x$$

and

$$x_{2^{n-1}+1} + \dots + x_{2^n} = y$$

in

$$2^n f((x_1 + x_2 + \dots + x_{2^n}) / (2^n)) = \sum_{i=1}^{2^n} f(x_i)$$

then

$$2^n f((x+y)/2) = 2^{n-1}f(x) + 2^{n-1}f(y)$$

for all $x, y \in V$. So

$$2f((x+y)/2) = f(x) + f(y)$$

for all $x, y \in V$. Hence f is additive. The convergence is obvious.

2.4.2. Theorem

Let V and W to be two Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f: V \rightarrow W$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$. Assume that there exist constants $\varepsilon \geq 0$ and $p \in [0, 1)$ such that:

$$\|f((x+y)/2) - f(x/2) - f(y/2)\| \leq \varepsilon (\|x/2\|^p + \|y/2\|^p)$$

for any $x, y \in V$. Then there exist a unique \mathbb{R} -Linear mapping $T: V \rightarrow W$ such that :

$$\|f(x) - T(x)\| \leq \varepsilon \|x\|^p / (4(2-2p))$$

for any $x \in V$, where Hyers-Ulam stability constant is for Jensen's functional equation.

Proof

Set

$$1/2^n T(x) = \lim_{n \rightarrow \infty} 1/2^n [f(2^n x)]$$

and multiplying the assumed equation

$$\|f((x+y)/2) - f(x/2) - f(y/2)\| \leq \varepsilon (\|x/2\|^p + \|y/2\|^p)$$

to 2^{n+1} :

$$\begin{aligned} \|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| &\leq \varepsilon (\|2^n x\|^p + \|2^n y\|^p) \\ &= 2^{np} \varepsilon (\|x\|^p + \|y\|^p) \end{aligned}$$

Therefore

$$1/2^n \|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| \leq 2^{n(p-1)} \varepsilon (\|x\|^p + \|y\|^p)$$

Or

$$\begin{aligned} \lim_{n \rightarrow \infty} 1/2^n \|f[2^n(x+y)] - f[2^n(x)] - f[2^n(y)]\| &\leq \lim_{n \rightarrow \infty} 2^{n(p-1)} \varepsilon (\|x\|^p + \|y\|^p) \\ &= 0 \end{aligned}$$

Or

$$\left\| \lim_{n \rightarrow \infty} 1/2^n f[2^n(x+y)] - \lim_{n \rightarrow \infty} 1/2^n f[2^n(x)] - \lim_{n \rightarrow \infty} 1/2^n f[2^n(y)] \right\| = 0$$

Or

$$\|T((x+y)/2) - T(x/2) - T(y/2)\| = 0$$

For any $x, y \in V$.

To prove stability our claim that:

$$\|f(2^n x) / 2^n - f(x)\| \leq \varepsilon \|2x\|^p \sum_{m=0}^{n-1} 2^m (p-1)$$

For any integer n , and some $\varepsilon \geq 0$. The verification of above equation following by induction on n . Indeed the case $n = 1$ is clear because by the hypothesis we can find ε , that is greater or equal to zero, and p such that $0 \leq p < 1$ with:

$$\|f(2x)/2 - f(x)\| \leq \varepsilon \|2x\|^p$$

Assume now that

$$\|f(2^n x)/2^n - f(x)\| \leq \epsilon \|2x\|^p \sum_{m=0}^{n-1} 2^m (p^{-1})$$

holds and we want to prove it for the case $n + 1$. However this is true, because by

$$\|f(2^n x)/2^n - f(x)\| \leq \epsilon \|2x\|^p \sum_{m=0}^{n-1} 2^m (p^{-1})$$

We obtain

$$\|f(2^n \cdot 2x)/2^n - f(x)\| \leq \epsilon \|4x\|^p \sum_{m=0}^{n-1} 2^m (p^{-1})$$

Therefore

$$\|f(2^n \cdot 2x)/2^{n+1} - 1/2f(2x)\| \leq \epsilon \|2x\|^p \sum_{m=0}^{n-1} 2^m (p^{-1})$$

By the triangle inequality we obtain:

$$\|f(2^{n+1}x)/2^{n+1} - f(x)\| \leq \|1/2^{n+1} f(2^{n+1}x) - 1/2f(2x)\| + \|1/2f(2x) - f(x)\| \leq \epsilon \|2x\|^p \sum_{m=0}^{n-1} 2^m (p^{-1})$$

and is valid for any integer n . It follows then that

$$\|f(2^n x)/2^n - f(x)\| \leq \epsilon \|2x\|^p \cdot (2/(2-2p))$$

because $\sum_{m=0}^{\infty} 2^m (p^{-1})$ converges to $(2/(2-2p))$, as $0 \leq p < 1$ we obtain

$$\|T(x) - f(x)\| \leq \epsilon \|2x\|^p \cdot (2/(2-2p))$$

Hence

$$\|T(x) - f(x)\| \leq \epsilon \|x\|^p \cdot (4/(2-2))$$

which completes the proof.

References

- [1] T. Aoki, on the stability of the linear transformation in banach spaces, *J. Math. Soc. Japan*, 2(1950)64-66. J. Clerk Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed., vol. 2. Oxford: Clarendon, 1892, pp.68-73.
- [2] I D. H. Hyers. "on the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* 27(1941),222-224.
- [3] Z. Kominek, on a local stability of the jensen functional EQUATION, *Demonstratio Math.*, 22 (1989),499-507.
- [4] R TH.M. Rassias, on the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* 72(1978),297-300.
- [5] M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience, New York, NY, USA, 1960. 1 .