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A note on localization of supplemented modules

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Abstract: In this paper we study on commutative rings with identity and all modules are unital left R-modules unless otherwise stated. We define the concept of small submodules for localization modules and additionally, we present the relation between an R-module M and an R_p-localization module M_p for all maximal ideals of R in view of being supplemented.

Keywords: Small Submodule, Supplemented Module, Multiplicative Closed Set, Localization Module

1. Introduction

Throughout this paper R is a commutative ring with unity and all modules are unital left R-modules. $N \leq M$ means that N is a submodule of M. Any submodule S is called a small submodule of M if for every proper submodule A of M, $M \neq A + S$. We will use the notation $S \ll M$ to indicate that any submodule S is small in M. Let M be an R-module and N be a submodule of M. A supplement of M K of M minimal with respect to property $M = N + K$ equivalently, $M = N + K$ and $M \cap K \ll K$. M is called supplemented if every submodule of M has a supplement in M. Supplemented modules are useful in characterizing semiperfect and perfect rings [10].

Suppose that we have a commutative ring R with a prime ideal P of R. We first introduce an equivalence relation $\sim$ on ther set $R \times (R \backslash P)$, we write $(r,t) \sim (r',t')$, where $r, r' \in R$ and $t, t' \in R \backslash P$, if there exists $t'' \in R \backslash P$ such that $t''tr' = t''t'r$. This equivalence relation partitions the set $R \times (R \backslash P)$; the set of equivalence classes is denoted by R_p; the equivalence class to which the element (r,t) of $R \times (R \backslash P)$ belongs is denoted by $\frac{r}{t}$. By defining well-known addition and multiplivation mappings on R_p, the classical construction of the localization of a commutative ring is obtained at a prime ideal [14].

2. Rings and Modules

First let us recall some basic concepts of ring theory.

Definition 1.
A ring R is a non-empty set together with two binary operations $(a,b) \mapsto a + b$ and $(a,b) \mapsto ab$ (addition and multiplication, respectively), subject to the following conditions:

$i)$ the set R together with addition is an abelian group
$ii)$ $a(bc) = (ab)c$ for all $a, b, c \in R$
$iii)$ $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in R$.

Definition 2.
A ring R is said to be commutative if $ab = ba$ for all $a, b \in R$.

Definition 3.
An element of R is called a unity and is denoted by 1_R, if $1_R a = a 1_R = a$ for all $a \in R$.

Definition 4.
A non-zero element a in a ring R is said to be a left [resp. right] zero divisor if there exists a non-zero $b \in R$ such that $ab = 0$ [resp. $ba = 0$]. A zero divisor is an element of R which is both a left and a right zero divisor.

Definition 5.
A commutative ring R with identity and no zero divisors is called an integral domain.

Definition 6.
By a left ideal of a ring R we mean a nonempty subset A of R such that $a - b \in A$ and $ra \in A$ for all $a, b \in A$ and $r \in R$.

Similarly, by a right ideal of R we mean a nonempty subset B of R such that $a-b \in B$ and $ar \in B$ for all $a, b \in B$ and $r \in R$.

A left ideal of R which is at the same time a right ideal of R is called an ideal of R. If R is a commutative ring, then the left ideals, right ideals and ideals of R coincide. An ideal of R is called proper if it is different from R.

Definition 7.

An ideal P in a ring R is said to be prime if $P \neq R$ and for any ideals A, B in R, $AB \subset P \Longrightarrow A \subset P$ or $B \subset P$.

Proposition 1.

If P is an ideal in a ring R such that $P \neq R$ and for all $a, b \in R$ $ab \in P \Longrightarrow a \in P$ or $b \in P$, then P is prime.

Definition 8.

An ideal M in a ring R is said to be maximal if $M \neq R$ and for every ideal N such that $M \subset N \subset R$, either $N = M$ or $N = R$.

Remark 1.

Let R be a commutative ring with identity. Then every maximal ideal of R is prime.

Definition 9.

A non-empty subset S of a ring R is multiplicative (closed) provided that $1_R \in S$ and $ab \in S$ for all $a, b \in S$

Example 1.

The set S of all elements in a nonzero ring with identity that are not zero divisors is multiplicatively closed.

Remark 2.

It is clear that a proper ideal P of R is prime if and only if $R-P$ is multiplicatively closed.

2.1. Modules

Definition 10.

Let R be a ring, a (left) R-module is an additive abelian group A together with a function $R \times A \longrightarrow A$ (the image of (r, a) being denoted by ra) such that for all $r, s \in R$ and $a, b \in A$:

i) $r(a+b) = ra + rb$

ii) $(r+s)a = ra + sa$

iii) $r(sa) = (rs)a$

Definition 11.

Let R be a ring, A an R-module and B a non-empty set of A. B is a submodule of A provided that B is an additive subgroup of A and $rb \in B$ for all $r \in R$, $b \in B$.

Definition 12.

Let A and B be modules over a ring R. A function $f\colon A \longrightarrow B$ is an R-module homomorphism provided that for all $a, b \in A$ and $r \in R$:

$f(a+b) = f(a) + f(b)$ and $f(ra) = rf(a)$

Definition 13.

A submodule K of an R-module M is called superfluous or small in M written $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$.

Definition 14.

Let U be a submodule of the R-module M. A submodule $V \subseteq M$ is called a supplement of U in M if V is a minimal element in the set of submodules $L \subseteq M$ with $U + L = M$.

Proposition 2.

V is a supplement of U if and only if $U + V = M$ and $U \cap V$ is superfluous in V.

3. Localization of Modules

Definition 15.

Let R be a ring and M be an R-module. Let S be a multiplicatively closed set in R. Let T be the set of all ordered pairs (x, s) where $x \in M$ and $s \in S$. Define a relation on T by $(x, s) \sim (x', s')$ if there exists $t \in S$ such that $t(sx' - s'x) = 0$. This is an equivalence relation on T and we denote equivalence classes of (x, s) by "$\frac{x}{s}$". Let $S^{-1}M$ denote the set of equivalence classe of T with respect to this relation. We can make $S^{-1}M$ into an R-module by setting

$$\frac{x}{s} + \frac{y}{t} = \frac{tx + sy}{st}, \quad a\left(\frac{x}{s}\right) = \frac{ax}{s}$$

The R-module $S^{-1}M$ is called a quotient module or a module of quotients [2].

Remark 3.

If $0 \in S$, then $S^{-1}M = 0$. In our discussion of quotient modules we shall always assume that $0 \notin S$ [2].

Proposition 3.

[2], let R be a ring and S be a multiplicatively closed set in R. Let M be an R-module and N and K be submodules of M. Then,

1) $S^{-1}(N + K) = S^{-1}N + S^{-1}K$
2) $S^{-1}(N \cap K) = S^{-1}N \cap S^{-1}K$
3) $S^{-1}(NK) = (S^{-1}N)(S^{-1}K)$

Definition 16.

Let R be a commutative ring with identity, M be an R-module, and $S = R - P$ be a multiplicatively closed subset of R such that P is a prime ideal of R. Then the module of quotients $S^{-1}M$ is called the localization of M at P and is denoted M_P [5].

Proposition 4.

Let R be a ring, M be an R-module and A and B be submodules of M. Then $A = B$ if and only if $A_P = B_P$ for every maximal ideal P of R [5].

4. Localization of Supplemented Modules

Definition 17.

Let R be a ring, S be a multiplicatively closed subset of R, M be an R-module, $S^{-1}N$ be a submodule of the quotient module $S^{-1}M$. If $S^{-1}K + S^{-1}L = S^{-1}M$ implies $S^{-1}L = S^{-1}M$ for every submodule $S^{-1}L \subseteq S^{-1}M$ then the submodule $S^{-1}K$ of $S^{-1}M$ is called small in $S^{-1}M$.

4.1. Properties of Superfluous Submodules

Proposition 6.

Let $S^{-1}K, S^{-1}L, S^{-1}N$ and $S^{-1}M$ be $S^{-1}R$-modules.

1. If $S^{-1}f: S^{-1}M \to S^{-1}N$ and $S^{-1}g: S^{-1}N \to S^{-1}L$ are two epimorphisms, then $S^{-1}g_0S^{-1}f$ is superfluous if and only if $S^{-1}f$ and $S^{-1}g$ are superfluous.
2. If $S^{-1}K \subseteq S^{-1}L \subseteq S^{-1}M$, then $S^{-1}L \ll S^{-1}M$ if and only if $S^{-1}K \ll S^{-1}M$ and $S^{-1}L/_{S^{-1}K} \ll S^{-1}M/_{S^{-1}K}$.
3. If $S^{-1}K_1, S^{-1}K_2, \dots, S^{-1}K_n$ are superfluous modules of $S^{-1}M$, then $S^{-1}K_1 + S^{-1}K_2 + \dots + S^{-1}K_n$ is also superfluous in $S^{-1}M$.
4. For $S^{-1}K \ll S^{-1}M$ and $S^{-1}f: S^{-1}M \to S^{-1}N$ is a homomorphism we get $S^{-1}f(K) \ll S^{-1}N$.
5. If $S^{-1}K \subseteq S^{-1}L \subseteq S^{-1}M$, then $S^{-1}L$ is a direct summand in $S^{-1}M$, then $S^{-1}K \ll S^{-1}M$ if and only if $S^{-1}K \ll S^{-1}L$.

Proof.

It is easily seen dually to the proof of the properties of superfluous submodules given for an R-module M.

Definition 18.

Let $S^{-1}U$ and $S^{-1}V$ be submodules of quotient module $S^{-1}M$ where M is an R-module and S is a multiplicative closed subset of a ring R. $S^{-1}V$ is a supplement of $S^{-1}U$ in $S^{-1}M$ if and only if $S^{-1}U + S^{-1}V = S^{-1}M$ and $S^{-1}(U \cap V)$ is superfluous in $S^{-1}V$.

Definition 19.

An $S^{-1}R$-module $S^{-1}M$ is called supplemented if every submodule of $S^{-1}M$ has a supplement in $S^{-1}M$.

Proposition 7.

Let R be a ring, M be an R-module and A be a submodule of M. Then $A \ll M$ if and only if $A_P \ll M_P$ for every maximal ideal P of R.

Proof.

($\Rightarrow$) Suppose $A \ll M$ and $A_P + B_P = M_P$ for a submodule B_P of M_P. It is easy to see $(A+B)_P = M_P$ from proposition 3. So, the equality $A + B = M$ can be verified from proposition 4. Hence we can say $B = M$ since $A \ll M$. Therefore, $B_P = M_P$.

($\Leftarrow$) Straight forward.

Proposition 8.

Let R be a ring, M be an R-module, A be a submodule of M and P be an arbitrary ideal of R. Then there is a supplement of A in M if and only if there is a supplement of A_P in M_P.

Proof.

($\Rightarrow$) Suppose M is supplemented. Then, there is a submodule B of M such that $A + B = M$ and $A \cap B \ll B$. Therefore $(A+B)_P = M_P$ and finally $A_P + B_P = M_P$ is obtained. Also $(A \cap B)_P \ll M_P$ because of the previous proposition. Hence B_P is a supplement of A_P in M_P.

($\Leftarrow$) Straight forward.

Corollary 1.

Let R be a ring and M be an R-module. Then M is supplemented if and only if M_P is supplemented for all maximal ideal P of R.

Proposition 9.

Let R be a ring, $f: M \to N$ be an R-module homomorphism and $S \subseteq R$ be a multiplicative closed subset. Then $S^{-1}(f(M)) = S^{-1} f(S^{-1}M)$.

Proof.

Straight forward.

Proposition 9.

Let M be an R-module. If M is a supplemented module then the homomorphic image of M_P is a supplemented module for all maximal ideal P of R.

Proof.

Let $f: M \to N$ be an R-module homomorphism and $S^{-1}f: M_P \to N_P$ be an R_P-module homomorphism via $\frac{m}{s} \mapsto \frac{f(m)}{s}$. Since the homomorphic image of a supplemented module is supplemented $f(M)$ is also supplemented in this situation. So $f(M)_P$ is supplemented from corollary 1. Hence, $S^{-1} f(M_P)$ is supplemented because of the previous proposition.

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